Solving algebraic problems using trigonometry

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A lot of problems in mathematics can be solved in different ways, applying various methods. Sometimes, considering the problem from an unusual point of view, one comes up with an interesting, easier and more elegant solution. The following examples show how problems that seem purely algebraic can be solved by looking at them from a trigonometric point of view. The general idea in all of the examples is to substitute a number and/or expression from the given with a suitable trigonometric function, and use the properties of this function in a conveniently chosen domain. Usually we choose the function with some of the well-known trigonometric identities in mind.

Problem 1: Given 13 arbitrary distinct real numbers prove that for two of them, let's say a and b, the following inequalities hold:

$$0 < \frac{a-b}{1+ab} < 2-\sqrt{3}$$

<u>Observations:</u>

1. The expression $\frac{a-b}{1+ab}$ is very similar to the expression $\frac{\tan x - \tan y}{1+\tan x.\tan y}$ in (6) and, according to (6), equals tan (x-y).
2. $\tan\frac{\pi}{12} = \tan(\frac{\pi}{4}-\frac{\pi}{6}) = \frac{\tan\frac{\pi}{4} - \tan\frac{\pi}{6}}{1+\tan\frac{\pi}{4}.\tan\frac{\pi}{6}} = \frac{1-\frac{1}{\sqrt{3}}}{1+1.\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3}$. So, this is another expression, which can be connected to trigonometry.
3. Consider an interval of the length π and 13 distinct numbers in it. Obviously there will be at least two of them with a difference less than $\pi/12$.
4. We have arbitrary real numbers, and the tangent function takes all real values and is well defined in $(-\pi/2, \pi/2)$, and fortunately, this is an interval of length π.

So, here is <u>the solution</u>:

Let $x_1, x_2, x_3, \ldots x_{13}$ be 13 arbitrary real numbers. Obviously, there exist 13 numbers $y_1, y_2, \ldots. y_{13}$ from the interval $(-\pi/2, \pi/2)$, such that $x_k = \tan y_k$, $k=1,2,\ldots 13$.

By the Pigeonhole principle, there will be two numbers y_i and y_j, such that $0< y_i - y_j < \pi/12$. The tangent function is an increasing function in $(-\pi/2, \pi/2)$.

Therefore, $0=\tan 0<\tan (y_i - y_j)<\tan \pi/12=2-\sqrt{3}$.

Let $a= x_i=\tan y_i$, $b= x_j=\tan y_j$. Since $\tan(y_i - y_j) = \frac{\tan y_i - \tan y_j}{1+\tan y_i.\tan y_j} = \frac{a-b}{1+ab}$, for the numbers a, b (which are two of our 13 given real numbers) $0<\frac{a-b}{1+ab} < 2-\sqrt{3}$. Q.E.D.

Problem 2: Prove that among any four distinct positive numbers we can find two, let say a and b, such that $0 < \frac{a-b}{1+a+b+2ab} < 2-\sqrt{3}$.

Solution:

In the required inequality $2-\sqrt{3}= \tan \pi/12$. Thus, may be a tangent substitution will work well too. Following the same idea as in Problem 1, since we have only four numbers, the domain should be of the length $3.(\pi/12)= \pi/4$. Since the numbers are positive, the domain should be where the tangent function takes positive values. To come up with a suitable trigonometric identity, we will make some transformations on the given rational expression. Dividing both the numerator and the denominator by $ab\neq 0$

$$\frac{a-b}{1+a+b+2ab} = \frac{\frac{1}{b}-\frac{1}{a}}{\frac{1}{ab}+\frac{1}{b}+\frac{1}{a}+2} = \frac{(1+\frac{1}{b})-(1+\frac{1}{a})}{1+(1+\frac{1}{a}).(1+\frac{1}{b})}$$

, which looks like the expression in (6) again.

So, the solution comes now easy:

Let the given positive numbers be x_1, x_2, x_3, x_4 and the numbers a_1, a_2, a_3, a_4 be such that $\tan a_i = 1+\frac{1}{x_i}$, i=1,2,3,4. Since $x_i > 0$, the numbers a_i exist and belong to $(\pi/4, \pi/2)$. By the Pigeonhole principle, there must be two of the numbers a_i, let say a_k and a_p, such that $0 < a_k - a_p < \pi/12$. Therefore, $0=\tan 0<\tan (a_k - a_p)<\tan \pi/12=2-\sqrt{3}$, or $0 < \frac{\tan a_k - \tan a_p}{1+\tan a_k . \tan a_p} < 2-\sqrt{3}$

Let denote the respective x_k with b, x_p - with a. Hence,

$$\tan a_k = 1+\frac{1}{x_k} = 1+\frac{1}{b},$$
$$\tan a_p = 1+\frac{1}{x_p} = 1+\frac{1}{a}$$

. Then,

$$0 < \frac{(1+\frac{1}{b})-(1+\frac{1}{a})}{1+(1+\frac{1}{a}).(1+\frac{1}{b})} < 2-\sqrt{3} \Leftrightarrow 0 < \frac{a-b}{1+a+b+2ab} < 2-\sqrt{3}.$$

Q.E.D.

Problem 3: The sequence $x_0, x_1, x_2, x_3, \ldots x_n, \ldots$ is defined by:

$x_0=a$, $x_{n+1} = \dfrac{x_n\sqrt{3}+1}{\sqrt{3}-x_n}$ (a is a real number). Calculate x_{1994}.

(National Math Contest "Atanas Radev", 1994, Bulgaria)

<u>Solution:</u>

The expression in the recurrent formula is very similar to the trigonometric expression in (8). Hence, it is reasonable to perform the substitution: $x_n=\cot a_n$, n=0,1,2,…. (note that since the cotangent function takes all real values we do not state any restrictions on the terms of the given sequence).

Then $$x_{n+1} = \frac{x_n\sqrt{3}+1}{\sqrt{3}-x_n} = \frac{\cot a_n . \cot\frac{\pi}{6}+1}{\cot\frac{\pi}{6}-\cot a_n} = \cot(a_n - \frac{\pi}{6}) \qquad (*)$$

$$\Rightarrow a_{n+1} = a_n - \frac{\pi}{6}$$

Consider the sequence $a_0, a_1, \ldots, a_n, \ldots$.

It is defined by $a_0=\cot^{-1}a$, $a_{n+1} = a_n - \frac{\pi}{6}$, n=0,1,2,… (because of (*)) .

Hence, $x_{n+6} = \cot a_{n+6} = \cot(a_n - 6.\frac{\pi}{6}) = \cot(a_n - \pi) = \cot a_n = x_n$ (here we have used that the cotangent function is a periodic with a period π). Thus, the values of x_n repeat after every 6 terms. Since $1994 \equiv 2 \pmod 6$, $x_{1994}=x_2$.

Let calculate x_2.

$x_0=a$

$$x_1 = \frac{a\sqrt{3}+1}{\sqrt{3}-a}$$

$$x_2 = \frac{\frac{a\sqrt{3}+1}{\sqrt{3}-a}.\sqrt{3}+1}{\sqrt{3}-\frac{a\sqrt{3}+1}{\sqrt{3}-a}} = \frac{2a+2\sqrt{3}}{2-2a\sqrt{3}} = \frac{a+\sqrt{3}}{1-a\sqrt{3}}$$

Finally, $x_{1994} = \dfrac{a+\sqrt{3}}{1-a\sqrt{3}}$.

Problem 4: The sequences $\{a_n\}$ and $\{b_n\}$ are given by :

$$a_1 = 1,$$

$$a_{n+1} = a_n + \sqrt{a_n^{\ 2} + 1} \qquad (*)$$

$$b_n = \frac{a_n}{2^n}, n = 1,2,3,....$$

Prove that $\{b_n\}$ is a convergent sequence and find its limit.

<u>Solution:</u>

Obviously, $a_n > 0$. So , we can substitute $a_n = \cot \alpha_n$, $\alpha_n \in (0, \frac{\pi}{2})$, n=1,2,…Then, by (*)

$$\cot \alpha_{n+1} = \cot \alpha_n + \sqrt{\cot^2 \alpha_n + 1} = \frac{\cos \alpha_n}{\sin \alpha_n} + \sqrt{\frac{\cos^2 \alpha_n}{\sin^2 \alpha_n} + 1} = \frac{\cos \alpha_n}{\sin \alpha_n} + \frac{1}{|\sin \alpha_n|} =$$

$$(\alpha_n \in (0, \frac{\pi}{2}) \Rightarrow |\sin \alpha_n| = \sin \alpha_n)$$

$$= \frac{\cos \alpha_n + 1}{\sin \alpha_n} = \frac{2\cos^2 \frac{\alpha_n}{2}}{2\sin \frac{\alpha_n}{2} \cos \frac{\alpha_n}{2}} = \cot \frac{\alpha_n}{2}$$

Thus, the sequence $\{\alpha_n\}$ is defined by

$$\alpha_1 = \cot^{-1} 1 = \frac{\pi}{4} = \frac{\pi}{2^2}, \quad n=1,2,3,….$$

$$\alpha_n = \frac{\alpha_{n-1}}{2} = \frac{\pi}{2^{n+1}}$$

and, hence, $a_n = \cot \frac{\pi}{2^{n+1}}$, n=1,2,3,….. Now,

$$b_n = \frac{a_n}{2^n} = \frac{\cot \frac{\pi}{2^{n+1}}}{2^n} = \frac{\cos \frac{\pi}{2^{n+1}}}{\sin \frac{\pi}{2^{n+1}} . 2^n} = (\cos \frac{\pi}{2^{n+1}}) . \frac{\frac{\pi}{2^{n+1}}}{\sin \frac{\pi}{2^{n+1}}} . \frac{2}{\pi} .$$

Since

$$\frac{\sin x}{x} \xrightarrow[x \to 0]{} 0, \frac{\pi}{2^{n+1}} \xrightarrow[n \to \infty]{} 0,$$

$$\lim b_n = \lim (\cos \frac{\pi}{2^{n+1}}) . \lim (\frac{\frac{\pi}{2^{n+1}}}{\sin \frac{\pi}{2^{n+1}}}) . \lim (\frac{2}{\pi}), n \to \infty,$$

$$\lim b_n = \cos 0 . 1 . \frac{2}{\pi} = \frac{2}{\pi}$$

Therefore, $\{b_n\}$ is a convergent sequence and its limit is $2/\pi$ when $n \to \infty$.

Problem 5: Let the set M contain all numbers of the form $\frac{m+n}{\sqrt{m^2+n^2}}$, where m, n are natural numbers. Prove that for any two elements x<y of M there exists an element z∈M such that x<z<y.

<u>Solution:</u>

Let $x = \frac{m+n}{\sqrt{m^2+n^2}} \in M$. WOLOG, m≤n. Thus, there exists a∈(0, π/4), so that $\tan a = \frac{m}{n}$.

Using this substitution, we get consequently:

$$x = \frac{m+n}{\sqrt{m^2+n^2}} = \frac{m}{\sqrt{m^2+n^2}} + \frac{n}{\sqrt{m^2+n^2}} = \frac{1}{\sqrt{1+\frac{n^2}{m^2}}} + \frac{1}{\sqrt{\frac{m^2}{n^2}+1}} = \frac{1}{\sqrt{1+\cot^2 a}} + \frac{1}{\sqrt{\tan^2 a+1}} =$$

$$= \sin a + \cos a = \sqrt{2}(\sin a.\frac{1}{\sqrt{2}} + \cos a.\frac{1}{\sqrt{2}}) = \sqrt{2}(\sin a.\cos\frac{p}{4} + \cos a.\sin\frac{p}{4}) = \sqrt{2}.\sin(a+\frac{p}{4})$$

So, for every x∈M there exists a∈(0, π/4), such that $x = \sqrt{2}.\sin(a+\frac{p}{4})$ (**).

It is also true that if a∈(0, π/4) and tan a is a rational number ($\tan a = \frac{m}{n}$) there exists x∈M, such that $x = \sin a + \cos a = \sqrt{2}.\sin(a+\frac{p}{4})$ (just follow the above transformations backwards) (***).

Now, let x<y, x∈M, y∈M. By (**), there are two numbers a, b∈(0, π/4), such that

$$x = \sqrt{2}.\sin(a+\frac{p}{4})$$

$$y = \sqrt{2}.\sin(b+\frac{p}{4}).$$

Since the sin function is increasing in (0, π/2), from x<y ⇒a<b.

Let c∈(a,b), so that tan c is a rational number. By (***), there is a number z∈M, such that

$$z = \sqrt{2}.\sin(c+\frac{p}{4}).$$

Thus, since a<c<b⇒

$$a+\frac{p}{4} < c+\frac{p}{4} < b+\frac{p}{4} \Rightarrow \sqrt{2}.\sin(a+\frac{p}{4}) < \sqrt{2}.\sin(c+\frac{p}{4}) < \sqrt{2}.\sin(b+\frac{p}{4}) \Rightarrow$$

$$x < z < y$$

It proves the required.

Problem 6: Let x,y,z be real numbers, and none of them equals ±√3/3. Let also x+y+z=xyz. Prove the identity:

$$\frac{3x-x^3}{1-3x^2}+\frac{3y-y^3}{1-3y^2}+\frac{3z-z^3}{1-3z^2}=\frac{3x-x^3}{1-3x^2}.\frac{3y-y^3}{1-3y^2}.\frac{3z-z^3}{1-3z^2}.\ (****)$$

Solution:

Remember that $\tan 3x=\frac{3\tan x-\tan^3 x}{1-3\tan^2 x}$. (see the trigonometric identity (17)).

Substitute the numbers x,y,z as follows:

x=tan a, y=tan b, z=tan c, where a,b,c∈(-π/2, π/2), and none of them equals ±π/6.

Thus, (****) becomes

$\tan 3a+\tan 3b+\tan 3c=\tan 3a.\tan 3b.\tan 3c$ and must be proven given that

$\tan a+\tan b+\tan c=\tan a.\tan b.\tan c$.

In general, let us try to answer the question: Which conditions for u,v,w ensure that the following identity is true: $\tan u+\tan v+\tan w=\tan u.\tan v.\tan w$? Using some of the trigonometric identities and equivalent transformations, we get:

$$\tan u+\tan v+\tan w-\tan u.\tan v.\tan w=\frac{\sin u}{\cos u}+\frac{\sin v}{\cos v}+\frac{\sin w}{\cos w}-\frac{\sin u}{\cos u}.\frac{\sin v}{\cos v}.\frac{\sin w}{\cos w}=$$

$$=\frac{\cos w(\sin u.\cos v+\sin v.\cos u)+\sin w(\cos u.\cos v-\sin u\sin v)}{\cos u.\cos v.\cos w}=$$

$$=\frac{\cos w(\sin(u+v))+\sin w(\cos(u+v)}{\cos u.\cos v.\cos w}=\frac{\sin(u+v+w)}{\cos u.\cos v.\cos w}$$

The RHS is 0 if and only if u+v+w=kπ, for any integer number k. So, the desired condition is u+v+w=kπ.

Therefore, since from the given $\tan a+\tan b+\tan c=\tan a.\tan b.\tan c$,

(or $\tan a+\tan b+\tan c-\tan a.\tan b.\tan c=0$), it must be true that a+b+c=kπ. Then 3a+3b+3c= kπ. Hence $\tan 3a+\tan 3b+\tan 3c-\tan 3a.\tan 3b.\tan 3c=0$, which proves the given conditional identity.

Problem 7: Solve the inequality:

$$10^{3(x-1)}-3.10^{x-1}>\sqrt{3}$$

Solution:

Substitute $10^{x-1}=2y$, y>0

Therefore,

$$10^{3(x-1)}-3.10^{x-1}>\sqrt{3}\Leftrightarrow(2y)^3-3y>\sqrt{3}\Leftrightarrow 8y^3-6y-\sqrt{3}>0\Leftrightarrow 4y^3-3y-\frac{\sqrt{3}}{2}>0.$$

We will find the zeroes of the polynomial $f(y)=4y^3-3y-\frac{\sqrt{3}}{2}$. Looking at the trigonometric formula (18): $\cos 3x=4\cos^3 x-3\cos x$, we come up with an another substitution: y=cosφ, which changes the polynomial to $4\cos^3\varphi - 3\cos\varphi - \frac{\sqrt{3}}{2}$. To find its zeroes, we have to solve the trigonometric equation:

$4\cos^3\varphi - 3\cos\varphi - \frac{\sqrt{3}}{2} = 0 \Leftrightarrow \cos 3\varphi - \cos\frac{\pi}{6} = 0 \Leftrightarrow 3\varphi = \pm\frac{\pi}{6} + 2k\pi, k \in Z$. Consider $\varphi_1 = \frac{\pi}{18}$, $\varphi_2 = \frac{13\pi}{18}$, $\varphi_3 = \frac{25\pi}{18}$. Since f(y) is a polynomial of the third degree, the only three solutions of f(y)=0 are $y_1 = \cos\frac{\pi}{18}, y_2 = \cos\frac{13\pi}{18}, y_3 = \cos\frac{25\pi}{18}$. So, f(y)=4(y-$y_1$)(y-$y_2$)(y-$y_3$) and we are to solve the system of inequalities:

(1) 4(y-y_1)(y-y_2)(y-y_3)>0

(2) y>0

Note that y_2<0, y_3<0. So, the solutions of the system are all $y > \cos\frac{\pi}{18}$.

Therefore, $10^{x-1} > 2.\cos\frac{\pi}{18} \Leftrightarrow x > \log\left(2.\cos\frac{\pi}{18}\right) + 1$.

In conclusion, it is always a good idea to keep in mind approaching problems using methods, which do not belong to the respective math area. Sometimes this allows us to find more interesting and beautiful solutions even if the standard methods work well. For some problems, looking at them from a different angle may be your only chance to solve them. So, be creative and try as many ideas as you can.

The latest example of how this "interdisciplinary" approach works is the problem #6 of IMO 2001:
Let a,b,c,d be integers with a>b>c>d>0. Suppose that ac+bd=(b+d+a-c)(b+d-a+c). Prove that ab+cd is not prime.

Solution:
Since a,b,c,d>0, they may be considered as lengths of segments. The given equality ac+bd=(b+d+a-c)(b+d-a+c) is equivalent to $a^2 - ac + c^2 = b^2 + bd + d^2$ (1)
Suppose ABCD is a quadrilateral such that AB=a, BC=d, CD=b, AD=c, ∠BAC=60°, ∠BCD=120°. According to (1) and the Cosine Law such a quadrilateral exists, and $BD^2 = a^2 - ac + c^2 = b^2 + bd + d^2$ (2). Now, let ∠ABC=x, then ∠ADC=180°-x. The Cosine Law for the triangles ABC and ACD gives:

$AC^2 = a^2 + d^2 - 2ad\cos x = b^2 + c^2 + 2bc\cos x \Rightarrow 2\cos x = \frac{a^2 + d^2 - b^2 - c^2}{ad + bc}$. Thus,

$AC^2 = a^2 + d^2 - ad\frac{a^2 + d^2 - b^2 - c^2}{ad + bc} = \frac{(ab + cd)(ac + bd)}{ad + bc}$ (3)

Note that ABCD is a cyclic quadrilateral. Hence, by the Ptolemy's Theorem AC.BD=ab+cd, which, together with (2) and (3) gives:

$$AC^2.BD^2 = \frac{(ab+cd)(ac+bd)}{ad+bc}.(a^2 - ac + c^2) = ab + cd \Leftrightarrow \qquad (4)$$

$$(ac+bd)(a^2 - ac + c^2) = (ab+cd)(ad+bc)$$

Next, from (a-d)(b-c)>0⇒ab+cd>ac+bd, and from (a-b)(c-d)>0⇒ac+bd>ad+bc.
Therefore, ab+cd>ac+bd>ad+bc (5).
Assume that ab+cd is prime. Hence, from (5) (ab+cd, ac+bd)=1. Now, from (4) it follows that (ac+bd)/(ad+bc), but from (5) it is impossible.
It proves that ab+cd is not prime.
Remark: Example of four numbers (a,b,c,d) that satisfy the given conditions are (21,18,14,1) and (65,50, 34, 11)

Exercises

1. The sequence $x_0, x_1, \ldots x_n, ..$ is defined by

$x_0 = a,$

$$x_{n+1} = \frac{x_n^{\;2} - 1}{2x_n}, n = 0,1,2,....$$

Find a formula for x_n in terms of a and n.

2. Prove that among any four distinct real numbers in (0, 1) we can find two (let say a,b) so that $\sqrt{(1-a^2)(1-b^2)} > \frac{a}{2b} + \frac{b}{2a} - ab - \frac{1}{8ab}$ *(Czech Math Olympiad, 1994)*

3. The vertices of a regular n-sided polygon are coloured in several colours in such a way that the vertices coloured in the same colour are vertices of a regular polygon too. Prove that at least two of these polygons are congruent (have the same number of vertices). *(Math Olympiad USSR, 1970)*

4. Given a regular 2001-sided polygon $A_0A_1A_2\ldots A_{2000}$, inscribed in the unit circle. Prove that $|A_0A_1|.|A_0A_2|.|A_0A_3|\ldots|A_0A_{2000}| = 2001$.

5. 100 numbers are placed in the squares of a 10x10 square table. The sum of the numbers in each figure composed of one row and one column is at least 19. Find the least possible sum of all numbers in the table.

6. The numbers k and l are both natural. Find the smallest number of the form

(a) $|11^k - 5^l|$;

(b) $|36^k - 5^l|$;

(c) $|53^k - 37^l|$

7. On a circle, there is 1 red and 2001 white points. We consider the polygons with vertices at these points. Which polygons are more - those that have white vertices only, or those that have a red vertex? What is the difference between the numbers of the polygons of both types?

Notes:

(N1) Problems 1 and 2 are a direct application of the ideas from the talk. Find an appropriate trigonometric substitution.

(N2) Problems 3 and 4 can be solved using the idea that the vertices of the n-sided regular polygon can be regarded as n complex numbers that are the roots of the equation $x^n-1=0$.

For those who are not familiar with this matter, for any natural number n the above equation has n distinct complex roots $e_0, e_1, e_2, \ldots e_{n-1}$*, given by the expressions:*

$e_j = \cos \frac{2pj}{n} + i \sin \frac{2pj}{n}$, $j = 0,1,2,3,..,n-1$. *It is clear that ½*e_j*½=1 and if we denote with*

$e = \cos \frac{2p}{n} + i \sin \frac{2p}{n}$ *, it is true that*

(*) $e_j = e^j$ *for any natural j;*

(**) $e^k = e^l$ *for the natural numbers k, l if and only if k °l(mod n).*

Two of the other properties of these numbers are given in the

Lemma:

(1) $\sum_{j=0}^{n-1} e_j^k = 0$ *when k is not a multiple of n*

(2) $\sum_{j=0}^{n-1} e_j^k = n,$ *when k is a multiple of n.*

The proof follows directly from the formula for the sum of a geometric series, exponent laws, and (*) and (**).

(N3) For problems 5,6, and 7 you do not need any directions and help. Just have fun☺ and relax, solving them.

Reference:

1. *Iv. Simeonov, 'Applications of trigonometry in algebra', Matematika,1/1995 (in Bulgarian).*
2. *Vl. Barzov, 'Êîðåíè íà åäèíèöàòà', Matematika Plus, 3/2001,(in Bulgarian)*
3. ***Error! Bookmark not defined.***
4. *Iv. Tonov, D. Rakovska, T.Vitanov, K. Bankov, 'Bulgarian Mathematics Competitions - Selected problems', Regalia-6, 1998, Sofia*

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