GEOMETRIC INEQUALITIES: AN INTRODUCTION

CHARLES LEYTEM

The triangular inequality, Ptolemy's inequality

CONTENTS

1. Introduction 2
2. Triangular inequality 2
2.1. Triangular inequality 2
2.2. Basic inequalities 2
2.3. Problems 4
3. Ptolemy's inequality 6
3.1. Ptolemy's inequality 6
3.2. Fermat point 7
3.3. Lower bound for $u \cdot PA + v \cdot PB + w \cdot PC$ 7
3.4. Problems 8

1. Introduction

We introduce the two most basic geometric inequalities: The triangular inequality which is a measure of the collinearity of three points; its generalization, Ptolemy's inequality, which is a measure of the cocyclicity of four points.

2. Triangular inequality

2.1. **Triangular inequality.** Our starting point is Euclids Elements, proposition 19 and 20.

Proposition 2.1.1. *In any triangle the side opposite the larger angle is larger.*

As, in a right (obtuse) triangle, the right (obtuse) angle is largest we get:

Corollary 2.1.2. *In a right (obtuse) triangle the side opposite the right (obtuse) angle is largest.*

Proposition 2.1.3. *(Triangular inequality) In any triangle the sum of any two sides is larger than the remaining side.*

Proof. We show that in the triangle ABC we have $AB + AC > BC$. Prolong the side $[BA]$ past A up to D such that $AD = AC$. Then we have the following implications:

$$AD = AC \Rightarrow \angle ADC = \angle ACD \Rightarrow \angle BCD > \angle ADC \Rightarrow DB > BC$$

But, by construction, $DB = AC + AB$. □

2.2. **Basic inequalities.** Some useful inequalities can be deduced from the triangular inequality.

Lemma 2.2.1. *For a point P inside a triangle ABC, $BP + PC < BA + AC$.*

Proof. Prolong $[BP]$ to cut $[AC]$ in Q. By the triangular inequality:

$$BP + PC < BP + PQ + QC = BQ + QC < BA + AQ + QC = BA + AC$$

□

Lemma 2.2.2. *For a point P inside a convex quadrilateral $ABCD$, $AP+PD < AB+BC+CD$.*

Proof. Prolong $[AP]$ to cut $[CD]$ resp. $[BC]$ in Q.

In the first case

$$AP+PD < AP+PQ+QD = AQ+QD < AC+CQ+QD = AC+CD < AB+BC+CD$$

In the second case

$$AP+PD < AP+PQ+QD = AQ+QD < AQ+QC+CD < AB+BQ+QC+CD = AB+BC+CD$$

□

Lemma 2.2.3. *Suppose that $[BC]$ is the longest side of a triangle ABC. For points P and Q on the sides $[AB]$ and $[AC]$, $PQ \leq BC$.*

Proof. We consider different cases:

- If $\angle APQ$ is obtuse (or right) we have $PQ \leq AQ \leq AC \leq BC$.
- If $\angle BPQ$ is obtuse (or right) we have $PQ \leq BQ$.
  - If $\angle AQB$ is obtuse (or right) we have $BQ \leq AB \leq BC$.
  - If $\angle CQB$ is obtuse (or right) we have $BQ \leq BC$. Thus $PQ \leq BC$.

□

Corollary 2.2.4. *Suppose that $[BC]$ is the longest side of a triangle ABC. Suppose that, moreover, for points P and Q on the rays $[AB)$ and $[AC)$ and for $\lambda > 0$, we have $AP \geq \lambda AB$ and $AQ \geq \lambda AC$. Then we have $PQ \geq \lambda BC$.*

Proof. We can take P' on $[AP]$ and Q' on $[AQ]$ such that $AP' = \lambda AB$ and $AQ' = \lambda AC$. Then by Thales $P'Q' = \lambda BC$. As $\hat{A}$ is the largest angle in ABC, PQ is the longest side in triangle APQ, and we conclude by the preceding lemma. □

Lemma 2.2.5. *In a triangle ABC we have $AM < \frac{AB+AC}{2}$ where M denotes the midpoint of $[BC]$.*

Proof. Consider the parallelogram $ABDC$. The point M is the midpoint of its diagonals and thus $2AM = AD < AB + BD = AB + AC$. □

Lemma 2.2.6. *In a triangle ABC we have $AM < \max(AB, AC)$ where M denotes any point on $[BC]$.*

Proof. As one of $\angle AMB$ and $\angle AMC$ is obtuse, AM is shorter than the side opposite this obtuse angle.

This lemma also results from the preceding lemma. □

The following corollary is a restatement of the preceding lemma.

Corollary 2.2.7. *Let M be a variable point on $[BC]$. Then the AM considered as a function of M reaches its maximum on an extremity of $[BC]$.*

We are interested in maximizing $AM + BM + CM$. The fact that each summand reaches a maximum on an extremity of a given segment is not enough to imply that the sum reaches a maximum on an extremity of the segment. But the before established inequality $AM < \frac{AB+AC}{2}$ will imply this property.

Lemma 2.2.8. *Let M be a variable point on a segment $[DE]$ inside triangle ABC. Then the $AM + BM + CM$ considered as a function of M reaches its maximum on an extremity of $[DE]$.*

Proof. Let X, Y be points inside $[DE]$ such that M is the midpoint of $[XY]$. We know that $AM < \frac{AX+AY}{2}$, $BM < \frac{BX+BY}{2}$, $CM < \frac{CX+CY}{2}$. Therefore $AM+BM+CM < \frac{(AX+BX+CX)+(AY+BY+CY)}{2}$, and consequently one of $AX + BX + CX$, $AY + BY + CY$ is larger than $AM + BM + CM$. Therefore $AM + BM + CM$ can't reach its maximum inside $[DE]$. □

Proposition 2.2.9. *Denote s the semi-perimeter, p the perimeter and P a point of the triangle ABC. Then we have*

$$s < AP + BP + CP < p.$$

Proof. By the triangular equality we have

$$AB < AP + PB, \quad BC < BP + PC, \quad CA < CP + PA.$$

Adding these inequalities we get, after division by 2, $s < AP + BP + CP$.

By 2.2.1 we have

$$AP + PB < AC + CB, \quad BP + PC < BA + AC, \quad CP + PA < CB + BA.$$

Adding these inequalities we get, after division by 2, $AP + BP + CP < p$.

These two inequalities, being strict, can be improved in various ways. In the problem section we give examples of sharpening $AP + BP + CP < p$. In a later section we deal with the inequality $s < AP + BP + CP$. □

2.3. **Problems.**

Problem 2.3.1. *Prove that in a triangle the sum of the heights is less than the perimeter.*

Solution. With an obvious notation we have $h_a \leq b$, $h_b \leq c$, $h_c \leq a$. Add these inequalities to conclude.

Problem 2.3.2. *Show that the perpendicular bisector of a segment is the locus of points equidistant from the extremities of the segment.*

Solution. Call M the midpoint of the segment $[AB]$, and (m) its perpendicular bisector.

If $P \in (m)$, then the triangles AMP and BMP are congruent as they both have a right angle and sides adjacent to this angle congruent. Thus $AP = BP$.

Now suppose w.l.o.g. that P belongs to the open half-plane containing B. Call I the intersection point of (AP) and (m). Then, by the first part, $BI = AI$, and, by the triangular inequality $BI + IP > BP$. Thus $AP = AI + IP = BI + IP > BP$.

Problem 2.3.3. *For which point P inside a convex quadrilateral $ABCD$ is $AP+BP+CP+DP$ minimal?*

Solution. $AP + BP + CP + DP$ is minimal when P is the intersection point of the diagonals of the quadrilateral as $AP + PC \leq AC$ with equality iff P is on the diagonal AC, and similarly for the diagonal BD.

Problem 2.3.4. *Prove that in a convex quadrilateral $ABCD$*

$$\max(AB + CD, AD + BC) < AC + BD < AB + BC + CD + DA.$$

Solution. Denote I the intersection point of the diagonals of $ABCD$. Suppose $AB + CD$ is the largest sum. We have $AB < AI + IB$ and $CD < CI + ID$. Adding both inequalities we get $AB + CD < AI + IB + CI + ID = AC + BD$. Also $AC < \frac{1}{2}((AB + BC) + (CD + DA))$ and $BD < \frac{1}{2}((AB + AD) + (BC + CD))$. The second inequality follows.

Problem 2.3.5. *In a convex quadrilateral $ABCD$ we have*

$$AB + BD \leq AC + CD.$$

Prove that $AB < AC$.

Solution. By the first inequality of the preceding lemma we have $AB + CD < AC + BD$. Adding this inequality to the given one we get $2AB < 2AC \Leftrightarrow AB < AC$.

Problem 2.3.6. *For each pair of opposite sides of a cyclic quadrilateral take the larger length less the smaller length. Show that the sum of the two resulting differences is at least twice the difference in length of the diagonals.*

Solution. Denote the cyclic quadrilateral $ABCD$ and the intersection point of its diagonals I. Then $ABI \sim DCI$. Denote $\lambda \geq 1$ the positive coefficient of the similarity transforming DCI into ABI. Then $AB - CD = \lambda CD - CD = (\lambda - 1)CD$, and, using the triangular inequality,

$$(\lambda - 1)CD > (\lambda - 1)(CI - DI) = \lambda CI - CI - \lambda DI + DI = BI - CI - AI + DI = BD - AC$$

establishing the result.

We have seen that, if P is an interior point of triangle ABC, then

$$s < AP + BP + CP < p,$$

where p, resp. s denote the perimeter resp. semi-perimeter of ABC. The following problems are examples of sharpening inequality $AP + BP + CP < p$.

Problem 2.3.7. *The maximum of $AP+BP+CP$ is the sum of the two longest sides of the triangle.*

Solution. Suppose P is inside the triangle. By 2.2.8 applied to a segment containing P, $AM + BM + CM$ reaches its maximum on the edges of the triangle, and applying 2.2.8 once again to a segment containing P and lying inside an edge we conclude that $AM + BM + CM$ reaches its maximum on one of the vertices of the triangle. Thus the maximum is one of $a + b$, $b + c$, $a + c$.

Problem 2.3.8. *P is a point inside the acute triangle ABC. Show that*

$$\min(PA, PB, PC) + PA + PB + PC \leq AB + BC + CA.$$

Solution 1. The situation is slightly more complex than in the preceding problem, as we have three different expressions to maximize, depending on the location of P. As $PA = PB$, $PA = PC$, $PB = PC$ for P on a bisector of $[AB]$, $[AC]$, $[BC]$, the regions which determine the expression to maximize are delimited by the three bisectors of ABC. Thus we have to evaluate $\min(PA, PB, PC) + PA + PB + PC$ on the vertices of the three regions determined by the bisectors.

As ABC is acute these points are the vertices of the triangle, the midpoints of its sides and its circumcenter.

If P is a vertex we are done by the preceding exercise. Suppose P is the midpoint of e.g. $[AB]$.

- If $PC \geq \frac{c}{2}$, then $c + \frac{c}{2} + PC \leq c + \frac{a+b}{2} + \frac{a+b}{2} = a + b + c$.
- If $PC \leq \frac{c}{2}$, then $c + 2PC \leq c + 2\frac{a+b}{2} = a + b + c$.

It remains to establish the inequality $4R \leq a + b + c$ when P is at the circumcenter. But

$$4R \leq a + b + c \Leftrightarrow 2 \leq \sin \hat{A} + \sin \hat{B} + \sin \hat{C}$$

which is easily seen to be true. For example fix $\hat{C}$, write $\hat{B} = 180^0 - \hat{C} - \hat{A}$ and determine the minimum of $\sin \hat{A} + \sin \hat{B} + \sin \hat{C}$.

Solution 2. Consider the medial triangle $A'B'C'$ of ABC. It divides ABC into four regions, and each region is covered by at least two of the trapezoids $ABA'B'$, $BCB'C'$, $CAC'A'$.

Suppose for example that P belongs to $ABA'B'$ and $BCB'C'$. We have $\min(PA, PB, PC) + PA + PB + PC \leq PB + (PA + PB + PC) = (PA + PB) + (PB + PC)$. By lemma 2.2.2 we get $PA + PB < AB' + B'A' + A'B$ and $PB + PC < BC' + C'B' + B'C$. Adding these two inequalities we get $PB + (PA + PB + PC) < AB' + B'A' + A'B + BC' + C'B' + B'C = (A'B + C'B') + (AB' + B'C) + (BC' + B'A') = a + b + c$.

Problem 2.3.9. *If ABC is an acute triangle with circumcenter O, orthocenter H and circumradius R, show that for any point P on the segment $[OH]$,*

$$PA + PB + PC \leq 3R.$$

Solution. As before, $PA + PB + PC$ reaches its maximum on the extremities of $[OH]$. When $P = O$ we have equality. It remains to check for $P = H$. Call A', B', C' the midpoints of the sides of ABC. Then

$$HA + HB + HC = 2OA' + 2OB' + 2OC' = 2R(\cos \hat{A} + \cos \hat{B} + \cos \hat{C}) \leq 2R\frac{3}{2} = 3R.$$

3. Ptolemy's inequality

3.1. **Ptolemy's inequality.** We start with special case of Ptolemy's equality, interesting in its own right.

Proposition 3.1.1. *Let $ABCD$ denote a cyclic quadrilateral with ABC an equilateral triangle. Show that $BD = CD + AD$*

Proof. Erect an equilateral triangle ADE outwardly on side AD. Then ABD is congruent to ACE, as the angles $\angle BAD$ and $\angle CAE$ as well as the sides adjacent to these angles are equal. Therefore $BD = CE = CD + AD$. □

The same argument works when $ABCD$ is an arbitrary quadrilateral, except that, by the triangular inequality, $CE \le CD + DE = CD + AD$ with equality iff $\angle CDA$ is the supplement of $\angle ADE$, that is equal to 120^0.

Proposition 3.1.2. *Let $ABCD$ denote a convex quadrilateral with ABC an equilateral triangle. Then $BD \le CD + AD$ with equality iff $ABCD$ is cyclic.*

Proposition 3.1.3. *If $ABCD$ is a convex quadrilateral, then $AB \cdot CD + BC \cdot DA \ge AC \cdot BD$, with equality iff $ABCD$ is cyclic.*

Proof. The quadrilateral $ABCD$ is cyclic iff $\hat{B} = \pi - \hat{D}$.

We want to restate the condition for being cyclic in terms of a linearity condition. We draw a segment $[BE]$ such that $\angle EBA = \hat{D}$. Thus $ABCD$ is cyclic iff C, B, E are aligned.

To be able to use the triangular inequality, we have to relate moreover the lengths of the segments $[EB]$ and $[EC]$ to lengths in the quadrilateral. To this effect we choose $\angle BAE = \angle CAD$.

Then $ABE \sim ADC$ and we get:

$$\frac{EB}{CD} = \frac{AB}{AD} \Leftrightarrow EB = \frac{CD \cdot AB}{AD}.$$

Also, because $\angle EAC = \angle BAD$ and $\frac{EA}{AB} = \frac{CA}{AD}$, $EAC \sim BAD$. Therefore

$$\frac{EC}{BD} = \frac{AC}{AD} \Leftrightarrow EC = \frac{BD \cdot AC}{AD}.$$

By the triangular inequality we get

$$EB + BC \ge EC \Leftrightarrow \frac{CD \cdot AB}{AD} + BC \ge \frac{BD \cdot AC}{AD} \Leftrightarrow CD \cdot AB + BC \cdot AD \ge BD \cdot AC$$

with equality iff $ABCD$ is cyclic. □

A typical situation where Ptolemy applies is the special case given as an introductory example:

Corollary 3.1.4. *Let $ABCD$ denote a convex quadrilateral with ABC an equilateral triangle. Then $BD \le CD + AD$ with equality iff $ABCD$ is cyclic.*

Proof. By Ptolemy's theorem we have $AB \cdot CD + BC \cdot AD \ge AC \cdot BD$, and dividing by $AB = BC = AC$ we get $CD + AD \ge BD$. □

Here the quadrilateral $ABCD$ can be seen as a triangle ACD with an *equilateral* triangle erected on side $[AC]$.

Sometimes two or three triangles are erected on a central triangle. If these triangles are *isosceles* or *similar* to each other, then application of Ptolemy's inequality to the quadrilaterals formed by one of these triangles and a supplementary point inside the triangle can lead to interesting inequalities as seen in the following subsection.

We finish with a short comment on a quadrilateral $ABCD$ non-convex at D. On $[AC]$ erect a triangle ACD' with sides $AD' = AD$ and $CD' = CD$. Then $[AC]$ is the perpendicular bisector of $[DD']$. Apply Ptolemy's inequality to the convex quadrilateral $ABCD'$ to get $AB \cdot CD + BC \cdot AD = AB \cdot CD' + BC \cdot AD' \geq AC \cdot BD'$. Now, as B and D are on the same side of the bisector (AC) of $[DD']$, we have $BD < BD'$ implying $AB \cdot CD + BC \cdot AD > AC \cdot BD$. Thus Ptolemy's inequality is true for convex and non-convex quadrilaterals.

We could also draw parallels to (AD) and (DC) through A and C respectively, to obtain a convex quadrilateral $ABCI$ where I is the intersection point of these two parallels. As $ADCI$ is a parallelogram, $AD = CI$ and $DC = AI$. Applying Ptolemy's inequality we get $AB \cdot AD + BC \cdot DC = AB \cdot CI + BC \cdot AI \geq AC \cdot BI$. The inequality involves consecutive sides of the quadrilateral $ABCD$. This method used to turn a concave quadrilateral into a convex can also be very useful (cf. problem 3.4.4).

3.2. **Fermat point.** We want to improve the inequality $s < PA + PB + PC$ established in the preceding section. The situation is more complex because there is no obvious procedure to minimize $PA + PB + PC$.

Lemma 3.2.1. *Let $ABDC$ denote a convex quadrilateral with BDC an equilateral triangle and denote P a point inside ABC. Show that there is a unique point P such that $PA+PB+PC \geq AD$ with equality iff A, P, D are aligned and $\angle BPC = 120^0$.*

Proof. By corollary 3.1.4 we have $PB + PC \geq PD$ with equality iff $\angle BPC = 120^0$ i.e. P is on the circle circumscribed to BDC. Thus $PA + PB + PC \geq PA + PD = AD$ with equality iff A, P, D are aligned and $\angle BPC = 120^0$. The point P clearly exists and is unique. □

Erecting equilateral triangles $AC'B$, $BA'C$, $CB'A$ on the sides $[AB]$, $[BC]$, $[CA]$ of a triangle ABC with all angles smaller than 120^0, and applying the preceding lemma to the quadrilaterals defined by a point P internal to ABC and these triangles, we conclude:

Proposition 3.2.2. *In a triangle ABC with all angles smaller than 120^0 the circumcircles of $AC'B$, $BA'C$, $CB'A$ meet a a point P such that $\angle BPC = \angle CPA = \angle APB = 120^0$. Furthermore APA', BPB', CPC' are straight lines with $AA' = BB' = CC'$ and this common length is the minimal value of $PA + PB + PC$. The point P is called the Fermat point for the triangle ABC.*

3.3. **Lower bound for $u \cdot PA + v \cdot PB + w \cdot PC$.** We want to give a lower bound for $u \cdot PA + v \cdot PB + w \cdot PC$ where P is a point inside ABC and u, v, w are positive numbers satisfying the triangular inequality. Denote a, b, c the lengths of the sides of triangle ABC.

Define $x = \frac{u}{w}c$ and $y = \frac{v}{w}c$. Then x, y, c satisfy the triangular inequality. Erect a triangle ABC' with sides $AC' = y$ and $BC' = x$ outwardly on side $[AB]$. By Ptolemy's inequality we get $x \cdot PA + y \cdot PB \geq cPC'$. But $PC' \geq CC' - PC$ by the triangular inequality. Putting both inequalities together we deduce $x \cdot PA + y \cdot PB + c \cdot PC \geq c \cdot CC'$. Multiplying by $\frac{w}{c}$ we get $u \cdot PA + v \cdot PB + w \cdot PC \geq w \cdot CC'$.

In case $u = v = w$ we get the inequality leading to the Fermat point.

Another interesting case is given by $u = b$, $v = a$ and $w = c$. Here $ACBC'$ is a parallelogram and $CC' = 2m_c$. Thus $w \cdot CC' = 2c \cdot m_c \geq 4\mathcal{A} = 2p \cdot r$ and we get

$$PA \cdot b + PB \cdot a + PC \cdot c \geq 2p \cdot r.$$

Similarly

$$PB \cdot c + PC \cdot b + PA \cdot a \geq 2p \cdot r,$$

$$PC \cdot a + PA \cdot c + PB \cdot b \geq 2p \cdot r$$

Adding these three inequalities and factoring the left member we get:

$$(PA + PB + PC)(a + b + c) \geq 6p \cdot r.$$

This is equivalent to

$$PA + PB + PC \geq 6r.$$

Another approach to this inequality is given in problem 3.4.5.

3.4. **Problems.**

Problem 3.4.1. *For all positive reals a, b, c show that*

$$a\sqrt{b^2 - bc + c^2} + c\sqrt{a^2 - ab + b^2} \geq b\sqrt{a^2 + ac + c^2}.$$

Solution. Consider a quadrilateral $ABCD$ such that $\angle ADB = \angle BDC = 60^0$ and such that $DA = a$, $DB = b$, $DC = c$. Then

$$AB = \sqrt{a^2 - ab + b^2} \quad BC = \sqrt{b^2 - bc + c^2} \quad AC = \sqrt{a^2 + ac + c^2}.$$

Now apply Ptolemy's inequality to $ABCD$ to get $AB \cdot CD + AD \cdot BC \geq AC \cdot BD$.

Problem 3.4.2. *Let P be a point interior to the parallelogram $ABCD$ whose area is $\mathcal{A}$. Show that*

$$AP \cdot CP + BP \cdot DP \geq \mathcal{A}.$$

For which points P do we have equality?

Solution. Denote E, F, G, H the parallel projections of P onto the sides $[AB], [BC], [CD], [DA]$. Translate the triangle DPC by a vector $\overrightarrow{GE}$ to obtain a quadrilateral $APBG'$ with sides equal to AP, BP, CP, DP and diagonals AB and AD. We have $\mathcal{A} \leq AB \cdot AD$. Moreover Ptolemy's inequality for quadrilateral $APBG'$ gives:

$$AB \cdot AD = AB \cdot PG' \leq AP \cdot G'B + BP \cdot G'A = AP \cdot CP + BP \cdot DP.$$

Putting the two inequalities together we get the first part.

To have equality, we first need $ABCD$ to be a rectangle. Moreover Ptolemy's equality tells us that $EFGH$ has to be cyclic for the equality to be true. This is equivalent to $EFP \sim HPG$ and can be reformulated as

$$\frac{EP}{FP} = \frac{HP}{GP} \Leftrightarrow FP \cdot HP = EP \cdot GP \Leftrightarrow x(2a - x) = y(2b - y)$$

where a, b are the half side lengths of the rectangle and $x = FP$ and $y = EP$. Denoting $u = x - a$ and $v = y - b$ the last equation becomes $(a + u)(a - u) = (b + v)(b - v) \Leftrightarrow u^2 - v^2 = a^2 - b^2$ which is the equation of a equilateral hyperbola passing through A, B, C, D.

Problem 3.4.3. *$ABCDEF$ is a convex hexagon with $AB = BC$, $CD = DE$, $EF = FA$. Show that $\frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC} \geq \frac{3}{2}$. When does equality occur?*

Solution. We are facing here the more general situation mentioned in 3.1 when three isosceles, rather than equilateral, triangles are erected on a central triangle. Application of Ptolemy's theorem to the quadrilateral $ACEF$ gives

$$AC \cdot EF + CE \cdot AF \geq AE \cdot CF,$$

implying $\frac{FA}{FC} \geq \frac{c}{a+b}$, and similarly $\frac{DE}{DA} \geq \frac{b}{c+a}$, $\frac{BC}{BE} \geq \frac{a}{b+c}$ where $a = AC$, $b = CE$, $c = AE$. Thus

$$\frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC} \geq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

The last inequality follows easily after substituting $x = a + b$, $y = a + c$, $z = b + c$ as it reduces to

$$\frac{1}{2}\left(\frac{x+y-z}{z} + \frac{x+z-y}{y} + \frac{y+z-x}{x}\right) \geq \frac{3}{2} \Leftrightarrow \frac{1}{2}\left(\underbrace{\frac{x}{y} + \frac{y}{x}}_{\geq 2} + \underbrace{\frac{x}{z} + \frac{z}{x}}_{\geq 2} + \underbrace{\frac{y}{z} + \frac{z}{y}}_{\geq 2} - 3\right) \geq \frac{3}{2}.$$

To get equality we need to have $x = y = z \Leftrightarrow a = b = c$, i.e. the internal triangle is equilateral. As moreover the hexagon has to be cyclic for the three Ptolemy inequalities to become equalities, the hexagon has to be regular.

Problem 3.4.4. *Let D be a point inside an acute triangle ABC. Then*

$$DA \cdot DB \cdot AB + DB \cdot DC \cdot BC + DC \cdot DA \cdot CA \geq AB \cdot BC \cdot CA,$$

with equality iff D is the orthocenter of ABC.

Solution. As we have sums of products of three terms we apply Ptolemy's inequality twice. We can rewrite the left side as $DB \cdot (DA \cdot AB + DC \cdot BC) + DC \cdot DA \cdot CA$ and want to apply Ptolemy's inequality to $DA \cdot AB + DC \cdot BC$.

To this purpose we mark the point P such that $PC \parallel AD$ and $PC = AD$. Then $AP = DC$ in the parallelogram $ADCP$. Apply Ptolemy's inequality to the quadrilateral $ABCP$ to get

$$PC \cdot AB + AP \cdot BC \geq BP \cdot AC \Leftrightarrow DA \cdot AB + DC \cdot BC \geq BP \cdot AC.$$

To prove the initial inequality it remains to prove

$$DB \cdot BP \cdot AC + DC \cdot DA \cdot CA \geq AB \cdot BC \cdot CA \Leftrightarrow DB \cdot BP + DC \cdot DA \geq AB \cdot BC$$

Once more mark the point Q such that $QC \parallel BD$ and $QC = BD$. Then $BQ = DC$ in the parallelogram $BDCQ$. Apply Ptolemy's inequality to the quadrilateral $BPCQ$ to get

$$QC \cdot BP + BQ \cdot PC \geq PQ \cdot BC \Leftrightarrow BD \cdot BP + DC \cdot AD \geq PQ \cdot BC.$$

But $PQ = AB$ in the parallelogram $ABQP$, and the inequality is established.

We have equality iff both $ABCP$ and $BPCQ$ are cyclic, i.e. iff P and Q lie on the circumcircle of ABC. Therefore the parallelogram $APQB$ is also cyclic and is necessarily a rectangle. This implies $(CD) \perp (AB)$. As moreover $\angle CAD = \angle ACP = \angle ABP = \frac{\pi}{2} - \angle APB = \frac{\pi}{2} - \angle ACB$, we also have $(AD) \perp (BC)$. Finally D is the orthocenter of ABC.

Problem 3.4.5. *P is a point inside the triangle ABC. Starting from the construction of the Fermat point, show that*

$$PA + PB + PC \geq 6r,$$

where r is the inradius of ABC.

Solution. Denote BDC be the equilateral triangle erected outwardly on ABC. We know by the Fermat point proof that the minimum of $PA+PB+PC$ is AD. But
$AD^2 = AB^2 + BD^2 - 2AB\cdot BD\cos(\hat{B}+60^0) = c^2 + a^2 - 2ac\cos\hat{B}\cos 60^0 + 2ac\sin\hat{B}\sin 60^0$.
The cosine relation $b^2 = a^2 + c^2 - 2ac\cos\hat{B}$ and the area expression $S = \frac{1}{2}ac\sin\hat{B}$ give

$$AD^2 = c^2 + a^2 + (b^2 - a^2 - c^2)\frac{1}{2} + 4S\frac{\sqrt{3}}{2} = \frac{1}{2}(a^2 + b^2 + c^2 + 4\sqrt{3}S).$$

Thus the minimum is

$$\sqrt{\frac{1}{2}(a^2 + b^2 + c^2 + 4\sqrt{3}S)}.$$

Recall that $s \geq 3\sqrt{3}r$. For a proof write $s = (s-a)+(s-b)+(s-c) \geq 3((s-a)(s-b)(s-c))^{\frac{1}{3}}$ by the am-gm inequality. The right-hand side can be rewritten as

$$3(s(s-a)(s-b)(s-c))^{\frac{1}{3}}s^{-\frac{1}{3}} = 3S^{\frac{2}{3}}s^{-\frac{1}{3}} = 3(sr)^{\frac{2}{3}}s^{-\frac{1}{3}} = 3s^{\frac{1}{3}}r^{\frac{2}{3}},$$

leading to

$$s \geq 3s^{\frac{1}{3}}r^{\frac{2}{3}} \Leftrightarrow s^{\frac{2}{3}} \geq 3r^{\frac{2}{3}} \Leftrightarrow s \geq 3\sqrt{3}r.$$

Returning to our problem we get

$$a^2 + b^2 + c^2 \geq \frac{1}{3}(a+b+c)^2 = \frac{4}{3}s^2 \geq 36r^2$$

and

$$4\sqrt{3}S = 4\sqrt{3}sr \geq 36r^2.$$

Finally $AD \geq \sqrt{36r^2} = 6r$

Problem 3.4.6. *Let ABC be a triangle for which there exists an interior point F such that $\angle AFB = \angle BFC = \angle CFA$. Let the lines BF and CF meet the sides AC and AB at D and E respectively. Prove that*

$$AB + AC \geq 4DE$$

Solution. Construct equilateral triangles ACP and ABQ on the sides $[AC]$ and $[AB]$. The point F is the Fermat point of triangle ABC and is the intersection point of (BP) and (CQ). Let M denote the midpoint of $[AC]$, P' the point diametrically opposed to P on the circumcircle of ACP, and G the orthogonal projection of F onto AC. Then we have

$$\frac{PD}{DF} = \frac{PM}{FG} \geq \frac{PM}{MP'} = 3,$$

the last equality following from an easy calculation in the rectangular triangle PAP'.

Thus $PF \geq 4DF$ and similarly $QF \geq 4EF$. As $\angle PFQ = 120^0$ is obtuse, we get by corollary 2.2.4 that $PQ \geq 4ED$.

Finally $AB + AC = AQ + AP \geq PQ \geq 4ED$.