GEOMETRY

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Cyclic quadrilaterals, radical axis and inscribed circles

CONTENTS

1. Inscribed angles and cyclic quadrilaterals

1.1. **Basic notions.** Consider a circle Γ with center O. Let A, B, C, D be four points on Γ such that $D \notin A\overset{\frown}{C}B$. Recall the following notions:

(1) $\angle AOB$ is the angle at the center which intercepts the arc $A\overset{\frown}{C}B$.
(2) $\angle APB$ is an inscribed angle which intercepts the arc $A\overset{\frown}{C}B$ (the point P is on the arc $A\overset{\frown}{D}B$.
(3) $\angle BAT$ is a tangential angle which intercepts the arc $A\overset{\frown}{C}B$ (the point T is on the tangent at A to the circle Γ).

Proposition 1.1.1. *The inscribed angles which intercept $A\overset{\frown}{C}B$ all have same amplitude, equal to half of the amplitude of the angle at the center which intercepts the arc $A\overset{\frown}{C}B$. The tangential angle which intercepts the arc $A\overset{\frown}{C}B$ has the same amplitude as the inscribed angles.*

Definition 1.1.2. *A quadrilateral is called cyclic if its vertices are on a circle, the circum-circle of the quadrilateral.*

The following proposition results from proposition 1.1.1

Proposition 1.1.3. *A quadrilateral $ABCD$ is cyclic iff $\hat{A} + \hat{C} = \hat{B} + \hat{D} = 180^0$.*

As examples of cyclic quadrilaterals we have the rectangle and isosceles trapezium. Also, a non-rectangular parallelogram is not cyclic. We can reformulate proposition 1.1.3 as follows:

Proposition 1.1.4. *A quadrilateral $ABCD$ is cyclic iff $\angle ACB = \angle ADB$.*

Denote I the intersection points of the diagonals (AC) and (BD) and, if they exist, $J = (AB) \cap (CD)$, $K = (AD) \cap (BC)$ the intersection points of opposite sides. Using the theorem of Thales we can reformulate the property of being cyclic as follows:

Proposition 1.1.5. *A quadrilateral is cyclic iff one of the following holds:*

- $\frac{AB}{CD} = \frac{AI}{DI} = \frac{BI}{CI}$
- $\frac{AD}{BC} = \frac{AI}{BI} = \frac{DI}{CI}$
- $\frac{AB}{CD} = \frac{AK}{CK} = \frac{BK}{DK}$
- $\frac{AD}{BC} = \frac{AJ}{CJ} = \frac{DJ}{BJ}$
- $\frac{AC}{BD} = \frac{AK}{BK} = \frac{CK}{DK}$
- $\frac{AC}{BD} = \frac{AJ}{DJ} = \frac{CJ}{BJ}$

The following properties of cyclic quadrilaterals are very useful.

Proposition 1.1.6. *The area of a cyclic quadrilateral is given as*

$$[ABCD] = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

where a, b, c, d are the lengths of its sides and s is the semi-perimeter of the quadrilateral.

Proposition 1.1.7. *(Ptolemy inequality) For an arbitrary convex quadrilateral $ABCD$ we have*

$$AB \cdot CD + AD \cdot BC \geq BD \cdot AC$$

with equality iff the quadrilateral is cyclic.

1.2. **Problems.**

Problem 1.2.1. *In a square $ABCD$ a line cuts $[AB]$ in K and $[CD]$ in L. Let M be an arbitrary point interior to the segment $[KL]$. Denote N the second point of intersection of the circles circumscribed to AKM and LCM. Prove that A, N, K are collinear.*

Solution. We have to prove that $\angle CNM + \angle ANM = 180^0$. But, using the fact that the quadrilaterals $AKNM$ and $CLMN$ are cyclic and that $(AB) \parallel (CD)$, we get

$$\angle CNM = 180^0 - \angle CLM = 180^0 - \angle AKM = 180^0 - \angle ANM.$$

Problem 1.2.2. *Let $ABCD$ be a quadrilateral; let E, F be two points on $[BC]$ with E on $[BF]$. Moreover, suppose that $\angle BAE = \angle CDF$ and $\angle EAF = \angle FDE$. Prove that $\angle FAC = \angle EDB$.*

Solution. It is enough to prove that $ABCD$ is cyclic.
As $\angle EAF = \angle FDE$, the quadrilateral $ADFE$ is cyclic.
Therefore

$$\begin{aligned}\angle BAD + \angle BCD &= \angle BAF + \angle FAD + \angle BCD \\ &= \angle EDC + \angle FED + \angle BCD \\ &= \angle EDC + \angle CED + \angle ECD \\ &= 180^0\end{aligned}$$

implying that $ABCD$ is cyclic.

Problem 1.2.3. *Denote $[AB]$ a fixed diameter of a circle Γ. Let K denote a fixed point on $[AB]$, t the tangent to Γ at A, and $[CD]$ a chord passing through K.*
If $P = (BC) \cap t$ and $Q = (BD) \cap t$, prove that $AP \cdot AQ$ is constant if $[CD]$ varies.

Solution. As $[AB]$ is a diameter, the triangles ABC and ABD are rectangular. As $ACP \sim BCA$ and $ADQ \sim BDA$ we get

$$\frac{AP}{AC} = \frac{AB}{BC} \quad \frac{AQ}{AD} = \frac{AB}{BD}$$

implying $AP \cdot AQ = AB^2 \cdot \frac{AC}{BD} \cdot \frac{AD}{BC}$.
Again, as $BDK \sim CAK$ and $BCK \sim DAK$ we get

$$\frac{AC}{BD} = \frac{CK}{BK} \quad \frac{AD}{BC} = \frac{AK}{CK}$$

implying $AP \cdot AQ = AB^2 \cdot \frac{CK}{BK} \cdot \frac{AK}{CK} = AB^2 \cdot \frac{AK}{BK}$, a constant.

Problem 1.2.4. *(Brahmagupta) If a cyclic quadrilateral has perpendicular diagonals crossing at I, the line through I perpendicular to any side bisects the opposite side.*

Solution. Denote the cyclic quadrilateral $ABCD$ with the line (IH) perpendicular to $[BC]$ meeting $[DA]$ at K. Then

$$\angle DIK = \angle BIH = \angle ACB = \angle ADB = \angle KDI.$$

Hence the triangle KDI is isosceles. Similarly, so is the triangle KAI. Therefore $AK = KI = KD$.

Problem 1.2.5. *In the convex quadrilateral $ABCD$, the diagonals $[AC]$ and $[BD]$ are perpendicular and the opposite sides are non-parallel. We suppose that the intersection point P of the perpendicular bisectors of $[AB]$ and $[CD]$ is interior to $ABCD$. Show that $ABCD$ is cyclic iff the triangles ABP and CDP have equal area.*

Solution. If the quadrilateral is cyclic, then denoting $\angle ADB = \alpha$

$$[ABP] = \frac{1}{2}r^2\sin(2\alpha) = \frac{1}{2}r^2\sin(180^0 - 2\alpha) = [PCD]$$

as the diagonals are perpendicular and the angle at the center is double of the inscribed angle.

Reciprocally suppose $[ABP] = [PCD]$. If the quadrilateral $ABCD$ is not cyclic, we can assume that $PA = PB > PC = PD$. Now consider E, F on $(AC), (BD)$ such that $ABEF$ is cyclic. Call h, k the heights of triangles PCD, PEF with respect to $[CD], [EF]$. Then

$$[ABP] = [PEF] = \frac{EF \cdot k}{2} > \frac{CD \cdot h}{2} = [PCD]$$

as $EF > CD$ and $k > h$. This is a contradiction, implying that $ABCD$ must be cyclic.

Problem 1.2.6. *Consider a rectangle $ABCD$ with M an interior point. Prove that:*

$$[ABCD] \leq AM \cdot CM + BM \cdot DM$$

For which points M do we have equality?

Solution. Denote E, F, G, H the orthogonal projections of M onto the sides $[AB], [BC], [CD], [DA]$. Now apply Ptolemy's inequality to the quadrilateral $EFGH$:

$$[ABCD] = FH \cdot EG \leq EF \cdot GH + FG \cdot EH = AM \cdot CM + BM \cdot DM.$$

Ptolemy's equality tells us that $EFGH$ has to be cyclic for the equality to be true. This is equivalent to $EFM \sim HMG$ and, denoting $AB = 2a$, $BC = 2b$, we have

$$\frac{EM}{FM} = \frac{HM}{GM} \Leftrightarrow FM \cdot HM = EM \cdot GM \Leftrightarrow x(2a-x) = y(2b-y) \Leftrightarrow (x-a)^2-(y-b)^2 = a^2-b^2$$

showing that M belongs to an equilateral hyperbola passing through A, B, C, D.

2. Power of a point and radical axis

2.1. **Basic notions.** Consider a circle Γ and a point P; let A, B be the two, eventually coinciding, intersection points of Γ and a line (d) through P. The real $PA \cdot PB$ is independent of the line (d).

Definition 2.1.1. *The product $PA \cdot PB$ is called power of the point P with respect to the circle Γ and is denoted $\mathcal{P}_\Gamma(P)$.*

It is easy to check that the power of P with respect to Γ is given by $d^2 - r^2$ where d is the distance from P to the center of Γ and r is the radius of Γ.

The preceding definition leads to the notion of a radical axis of two circles. If the two circles intersect in two points, then by definition, all points on the line through these two intersection points have equal power with respect to each of the circles.

More generally we have the following:

Definition 2.1.2. *The set of all points which have equal power with respect to two given circles is a line called the radical axis of the two circles.*

Finally consider three circles and their three radical axes. If their centers are not aligned, two of these axes intersect in a point. This point has necessarely equal power with respect to the three circles. Therefore we can give the following definition:

Definition 2.1.3. *Consider three circles with centers not aligned. The intersection point of the radical axes of these circles is called radical center of the circle.*

2.2. **Problems.**

Problem 2.2.1. *Construct the radical axis of two non-intersecting circles and show that it is perpendicular to the line passing through the centers.*

Solution. It is obvious that the midpoints of a common tangent lie on the radical axis. Just draw the line through two such midpoints. By symmetry this line is perpendicular to the line passing through the centers.

Another construction makes use of the radical center of three circles. Consider a third circle which intersects both given circles. The radical center of these three circles lies on the radical axis of the two given circles. Repeating the construction with another circle we obtain two points of the radical axis of the initial circle.

Problem 2.2.2. *Consider a convex hexagon $ABCDEF$ such that*

$$AB = BC \quad CD = DE \quad EF = FA.$$

Show that the perpendiculars through A, C, E onto FB, BD, DF are concurrent.

Solution. Consider the circles with centers B, D, F and radii AB, CD, EF. Their radical axes pass through A, C, E and are perpendicular to FB, BD, DF. As they intersect in the radical center of the three circles we are done.

Problem 2.2.3. *Two circles Γ_1, Γ_2 intersect in M, N. Denote (l) the common tangent to Γ_1, Γ_2 such that M is closer to (l) than to N. The line (l) is tangent to Γ_1 at A and to Γ_2 at B. The parallel to (l) through M meets the circle Γ_1 again in C and the circle Γ_2 again in D. The lines CA and DB meet in E; the lines AN and CD meet in P; the lines BN and CD meet in Q.*

Show that $EP = EQ$.

Solution. Denote X the intersection point of MN (the radical axis) and (l). Then X is the midpoint of $[AB]$. It results immediately from Thales theorem that M is the midpoint of $[PQ]$.

To prove that $EP = EQ$ it is enough to show that $(EM) \perp (PQ)$. But

$$\angle BAE = \angle ACM = \angle AMC = \angle BAM$$

and

$$\angle EBA = \angle BDM = \angle BMD = \angle ABM.$$

Therefore M is symmetric to E with respect to the axis (AB) and $(EM) \perp (PQ)$.

Problem 2.2.4. *The in-circle of ABC touches $[BC], [CA], [AB]$ at D, E, F respectively. The point X inside ABC is such that the in-circle of XBC touches $[BC]$ at D also, and touches $[CX], [XB]$ at Y, Z respectively. Prove that $EFZY$ is a cyclic quadrilateral.*

Solution. If the quadrilateral $EFZY$ is cyclic, then the radical axes $(BC), (EF), (ZY)$ meet at the radical center.

Reciprocally, to establish that $EFZY$ is cyclic we have to show that $P = (BC) \cap (EF)$ and $Q = (BC) \cap (ZY)$ coincide. Now $\frac{BP}{CP} = \frac{FB}{CE}$ as $PBF \sim PEC$ and $\frac{BQ}{CQ} = \frac{ZB}{CY}$ as $QBZ \sim QYC$. As moreover $\frac{ZB}{CY} = \frac{FB}{CE}$, we get $\frac{BP}{CP} = \frac{BQ}{CQ}$, implying $P = Q$.

Finally $PE \cdot PF = PD^2 = PY \cdot PZ$ and $EFZY$ is cyclic.

3. Circles inscribed to triangles and quadrilaterals

3.1. **Triangles.** To solve in(ex)circle problems it is essential to know the lengths of the segments determined by the contact points of the in(ex)circles on the sides of the triangle and the symmetry of their disposition.

Call D, E, F the contact points of the in-circle on $[BC]$,$[CA]$ and $[AB]$ respectively. Similarly call D_a, E_a, F_a contact points of the ex-circle opposite to A on $[BC]$,$[AC)$ and $[AB)$ respectively. The following is an easy exercise.

Proposition 3.1.1. *The segments determined by the contact points of the in-circle (ex-circle) have the following lengths.*

$$DB = FB = s - b \quad DC = EC = s - c \quad EA = AF = s - a$$
$$AF_a = AE_a = s \quad BD_a = BF_a = s - c \quad CD_a = CE_a = s - b$$

3.2. **Quadrilaterals.** There is the following problem

Problem 3.2.1. *(Bulgarian Olympiad, third round, 1999) Let B_1 and C_1 be points on the sides $[AC]$ and $[AB]$ of triangle ABC. Lines (BB_1) and (CC_1) intersect at point D. Prove that a circle can be inscribed inside quadrilateral AB_1DC_1 if and only if the in-circles of the triangles ABD and ACD are tangent to each other.*

This problem reminds us of the following property:

Proposition 3.2.2. *A convex quadrilateral $ABCD$ has an in-circle iff $AB + CD = BC + DA$.*

Proof. The only if part is an easy exercise.

The if part is more delicate. Suppose that $AB + CD = BC + DA$. Now we can assume that $[BA)$ and $[CD)$ have a common point I. Then triangle IBC has an in-circle and we can draw a tangent to this circle passing through A and cutting $[CD)$ in D'. As this circle is inscribed in the quadrilateral $ABCD'$ we get $AB + CD' = BC + D'A$ by the first part. This together with the hypothesis implies that $DD' = |DA - D'A|$ which is only possible if triangle $DD'A$ is flat. Thus $D = D'$ and $ABCD$ has an inscribed circle. □

This proposition can be reformulated as follows:

Proposition 3.2.3. *A convex quadrilateral $ABCD$ has an in-circle iff the in-circles of triangles ABC and ACD are tangent to each other.*

Proof.

$$\begin{aligned} incircle(ABC) \text{ tangent to } incircle(ACD) &\Leftrightarrow \frac{AB + AC - BC}{2} = \frac{AD + AC - CD}{2} \\ &\Leftrightarrow AB - BC = AD - CD \\ &\Leftrightarrow AB + CD = AD + BC \end{aligned}$$

Thus we can apply the preceding proposition to conclude. □

Now we give a solution to the initial problem.

Solution.

$$\begin{aligned} incircle(ABD) \text{ tangent to } incircle(ACD) &\Leftrightarrow \frac{DA + DB - AB}{2} = \frac{DA + DC - AC}{2} \\ &\Leftrightarrow DB - AB = DC - AC \end{aligned}$$

The quadrilateral $ABDC$ is not convex, but we can repeat the argument used in proposition 3.2.2.

First suppose that a circle can be inscribed inside AB_1DC_1. Let it be tangent to sides $[AB_1]$, $[B_1D]$, $[DC_1]$, $[C_1A]$ at points E, F, G, H, respectively. We have

$$\begin{aligned} AB - DB &= (AH + HB) - (BF - FD) \\ &= (AH + BF) - (BF - FD) \\ &= AH + FD \\ AC - CD &= (AE + EC) - (CG - DG) \\ &= (AE + CG) - (CG - DG) \\ &= AE + DG \end{aligned}$$

implying $DB - AB = DC - AC$ and thus $incircle(ABD)$ is tangent to $incircle(ACD)$.

Conversely suppose that the two in-circles are tangent to each other. Triangle ABB_1 has an in-circle, and we can draw a tangent to this circle passing through D and cutting $[AB_1]$ in C'. By the preceding paragraph the in-circles of ABD and ADC' must be tangent to each other, implying $DB-AB = DC'-AC'$. As also $DB-AB = DC-AC$, we conclude $AC-AC' = DC-DC' \Leftrightarrow CC' = |DC - DC'|$. This implies that triangle DCC' is flat and therefore $C = C'$ implying that C_1D is tangent to the in-circle of triangle ABB_1. Thus AB_1DC_1 has an in-circle.

Problem 3.2.4. *In triangle ABC, the points D, E are on side $[BC]$ such that $\angle BAD = \angle CAE$. If M and N are the points of tangency of the in-circles of ABD and ACE with $[BC]$, then show that*

$$\frac{1}{MB} + \frac{1}{MD} = \frac{1}{NC} + \frac{1}{NE}$$

Solution. It is enough to show that the left-hand side depends only on the height of triangle ABD and on the angle $\angle BAD$.

We have

$$\begin{aligned} \frac{1}{MB} + \frac{1}{MD} &= \frac{1}{s-b} + \frac{1}{s-d} \\ &= \frac{2s-b-d}{(s-b)(s-d)} \\ &= \frac{a \cdot s(s-a)}{s(s-a)(s-b)(s-d)} \\ &= \frac{a \cdot h \cdot s(s-a)r}{h[ABD]^2 r} \\ &= \frac{2[ABD] \cdot \cot\frac{\alpha}{2} \cdot s \cdot r}{h[ABD]^2} \\ &= \frac{2\cot\frac{\alpha}{2}[ABD]}{h[ABD]} \\ &= \frac{2}{h}\cot\frac{\alpha}{2} \end{aligned}$$