Inscribed and circumscribed quadrilaterals

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If you don't know how to start with a geometric problem, try to find four points lying on a common circle. This advice is quite old, but still very useful in the IMO problems. It has been used since the very beginning of the history of the Mathematical Olympiads and is also efficitively applied nowadays. An observation that the four points are concyclic is often based on the following simple theorems comming from the school geometry.

Theorem 1.
Let $ABCD$ be a convex quadrilateral. Then the points A, B, C, D lie on a common circle if and only if $\angle ABC + \angle CDA = 180^\circ$ (fig. 1).

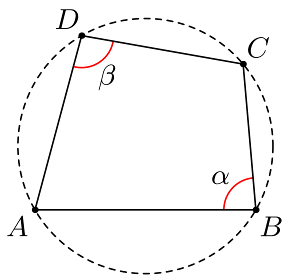

fig. 1

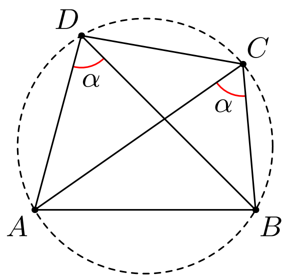

fig. 2

Theorem 2.
Let $ABCD$ be a convex quadrilateral. Then the points A, B, C, D lie on a common circle if and only if $\angle ACB = \angle ADB$ (fig. 2).

The following example is related to Problem 5 of the 1-st IMO (1959).

Example
Point E lies on the side BC of a square $ABCD$. Let $BFGE$ be a square lying outside of the square $ABCD$ (fig. 3). Prove that the lines AE, CF and DG intersect in a common point.

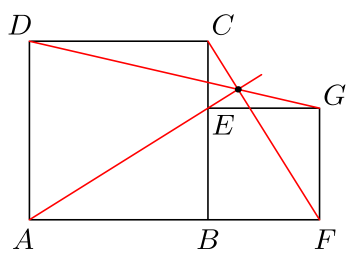

fig. 3

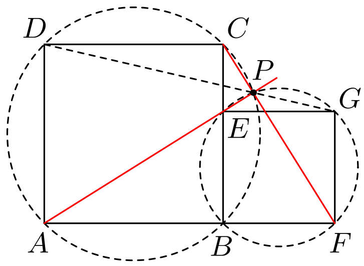

fig. 4

Solution
Let P be the intersection of the lines AE and CF (fig. 4). Our goal is to prove that the line DG passes through P. This will be achieved by observing that the measures of the angles DPA, APF and FPG sum up to 180°.

Note that from the equalities $AB = BC$, $\angle ABE = 90^\circ = \angle CBF$ and $BE = BF$ it follows that triangles ABE and CBF are congruent. Therefore $\angle BAP = \angle BCP$, which by Theorem 2 implies that the points A, B, C and P are concyclic. On the other hand, since $ABCD$ is a square, the circumcircle of triangle ABC passes through the point D. It means that the five points A, B, C, D and P lie on a common circle.

Therefore $\angle CPA = \angle CBA = 90^\circ$, which gives $\angle EPF = 90^\circ = \angle EGF$. Theorem 2 yields that the points E, P, G, F are concyclic. Since $EBFG$ is a square, the circumcircle of triangle

EFG passes through point B. It means that the five points E, B, F, G and P lie on a common circle. Hence we obtain

$$\angle DPA + \angle APF + \angle FPG = \angle DBA + \angle EPF + \angle FBG = 45^\circ + 90^\circ + 45^\circ = 180^\circ .$$

Now we consider dual configurations: circles inscribed in quadrilaterals. The following easy theorem is the key observation in many problems where tangents to a circle appear.

Theorem 3.

From a point P lying outside of a circle ω two tangents PA and PB are drawn (fig. 5). Then $PA = PB$.

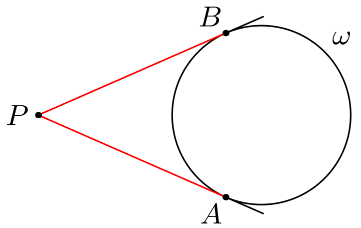

fig. 5

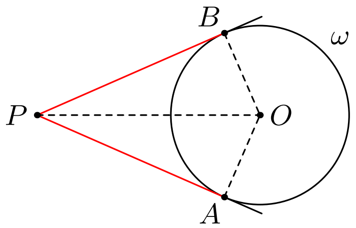

fig. 6

Proof

Let O be the center of ω (fig. 6). Then by the Pythagoras Theorem we get

$$PA^2 = PO^2 - OA^2 = PO^2 - OB^2 = PB^2 ,$$

implying $PA = PB$, which finishes the proof.

The next result is an easy, but useful application of Theorem 3.

Theorem 4.

Assume the incircle of triangle ABC is tangent to the sides BC, CA and AB at D, E and F, respectively. Set $a = BC$, $b = CA$, $c = AB$. Then we have

$$AE = AF = s - a , \quad BF = BD = s - b , \quad CD = CE = s - c ,$$

where s denotes the semiperimeter of triangle ABC.

Solution

Let $x = AE = AF$, $y = BF = BD$, $z = CD = CE$. Then we obtain the following system of equations:

$$\begin{cases} x + y = c \\ y + z = a \\ z + x = b \end{cases}$$

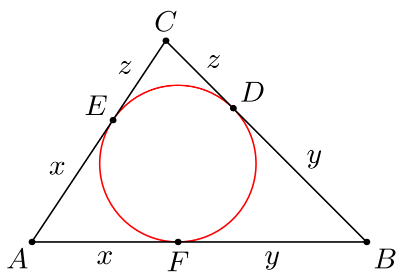

fig. 7

Solving this system we get $x = (b + c - a)/2$, $y = (c + a - b)/2$, $z = (a + b - c)/2$, which immidiately gives $x = s - a$, $y = s - b$, $z = s - c$.

Theorem 5.

Let $ABCD$ be a convex (fig. 8) or a concave (fig. 9) quadrilateral. Then there exists a circle inscribed in the quadrilateral $ABCD$ if and only if $AB + CD = BC + DA$.

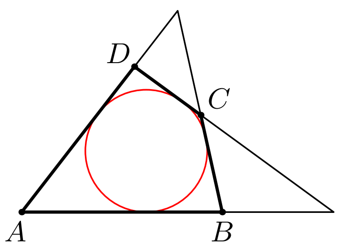

fig. 8

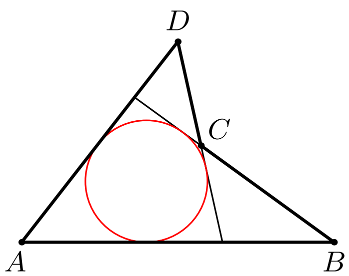

fig. 9

Proof

We present a proof for a concave quadrilateral $ABCD$. The proof for the convex case is very similar.

Assume first that there exists a circle inscribed in a concave quadrilateral $ABCD$ and let this circle be tangent to the lines AB, BC, CD, DA at K, L, M, N, respectively (fig. 10). Then applying Theorem 3 several times we obtain $AK = AN$, $BK = BL$, $DM = DN$ and $CM = CL$. Therefore

$$AB + CD = AK + BK + DM - CM = AN + BL + DN - CL = AD + BC\,.$$

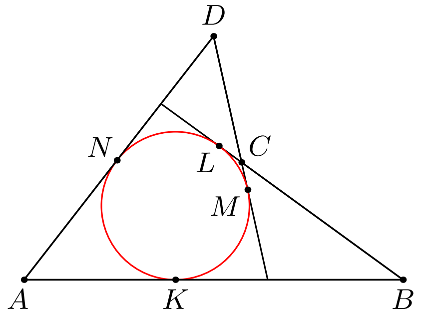

fig. 10

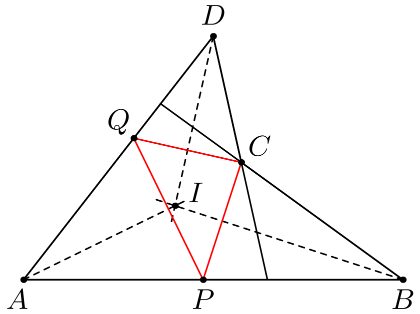

fig. 11

Conversly, assume that in a concave quadrilateral $ABCD$ it holds $AB + CD = BC + AD$. Without loss of generality assume that $AB > BC$. Then the last equality implies that $AD > CD$.

Let P be the point lying on the side AB such that $BP = BC$ (fig. 11). Similarly, let Q be the point lying on the side AD such that $QD = CD$. Then

$$AP = AB - BP = AB - BC = AD - CD = AD - DQ = AQ\,.$$

Therefore the angle bisectors of the angles ABC, CDA and DAB are the perpendicular bisectors of the sides of triangle CPQ and hence they intersect in a common point I. Moreover, the distances from the point I to all the lines AB, BC, CD, DA are equal, say r. Then the circle ω with center I and radius r lies inside the quadrilateral $ABCD$ and is tangent to all the lines AB, BC, CD, DA. Hence the circle ω is inscribed in a quadrilateral $ABCD$.

Example

Point P lies inside a triangle ABC. The lines AP, BP, CP intersect the sides BC, CA, AB at D, E, F, respectively. Prove that if it is possible to inscribe circles in quadrilaterals $AFPE$ and $BDPF$, then it is also possible to inscrible a circle in the quadrilateral $CDPE$.

Solution

The main idea is to note that the two given circles are inscribed in concave quadrilaterals $ABPC$ and $ABCP$, which by Theorem 5 implies that $AC + BP = AB + CP$ and $AB + CP = BC + AP$. Hence $AC + BP = BC + AP$, which using again Theorem 5 implies that it is possible to inscribe a circle in a concave quadrilateral $APBC$. Therefore there exists a circle inscribed in a (convex) quadrilateral $CDPE$.

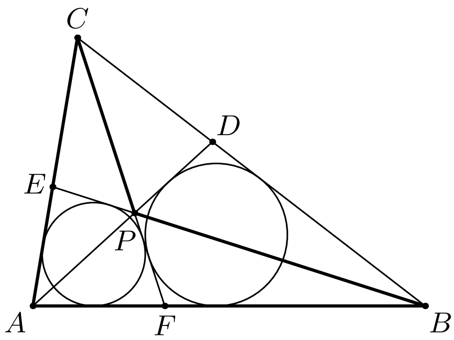

fig. 12

Problems

1. Let ABC be an acute-angled triangle with $\angle ACB = 60°$ (fig. 1). Points D and E are the feet of the perpendiculars from A and B to the lines BC and AC, respectively. Point M is the midpoint of the side AB. Prove that the triangle DEM is equilateral.

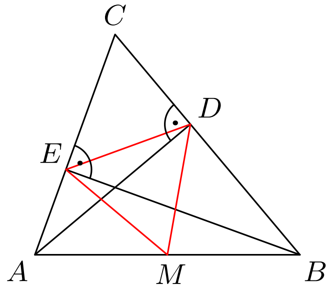

fig. 1

2. In an acute-angled triangle ABC points D, E, F are the feet of the altitudes taken from the vertices A, B, C, respectively (fig. 2).

(a) Prove that DA, EB, FC are the angle-bisectors of the angles of triangle DEF.

(b) Knowing that the measures of the angles of triangle ABC are equal 45°, 60°, 75°, determine measures of the angles of triangle DEF.

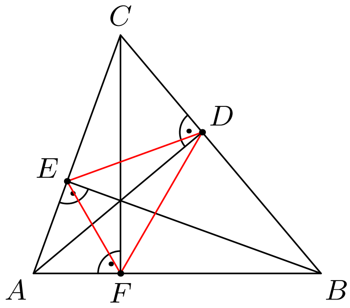

fig. 2

3. Let $ABCD$ be a square (fig. 3). Points E and F lie on the sides AB and AD, respectively, such that $\angle ECF = 45°$. The diagonal BD meets the lines CE and CF at P and Q, respectively. Prove that the points A, E, F, P and Q are concyclic.

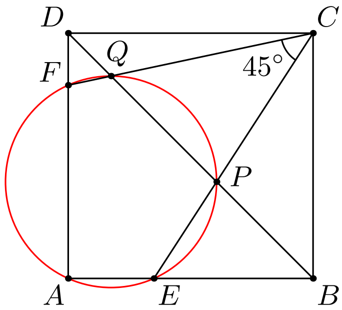

fig. 3

4. Given is an acute-angled triangle ABC. On sides BC and AC squares $BCFE$ and $ACGH$ are outwardly constructed (fig. 4). Prove that the lines AF, BG and EH intersect in a common point.

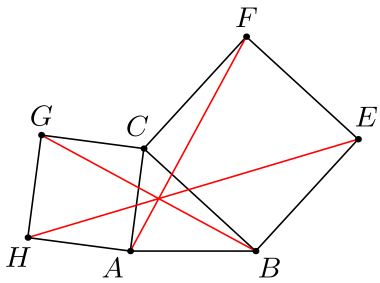

fig. 4

5. On sides BC, CA i AB of triangle ABC three equilateral triangles BCD, CAE i ABF are constructed outwardly. (fig. 5). Prove that:

(a) $AD = BE = CF$.

(b) The lines AD, BE and CF intersect in a common point.

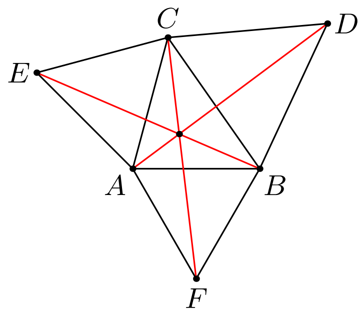

fig. 5

6. Point C lies on the line segment AB. Equilateral triangles BCD, CAE and ABF are constructed, as shown on figure 6. Prove that the lines AD, BE, CF intersect in a common point.

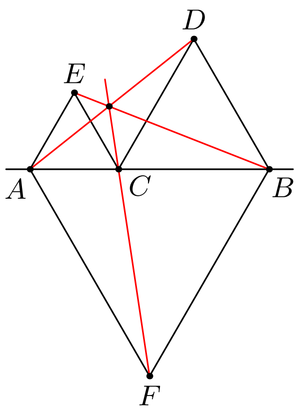

fig. 6

7. Let ABC be a triangle such that $AC = BC$ (fig. 7). Point M is the midpoint of the side AB. Point D lies on the line segment CM. Let K and L be the feet of the perpendiculars from D and C onto BC and AD, respectively. Prove that the points K, L and M are collinear.

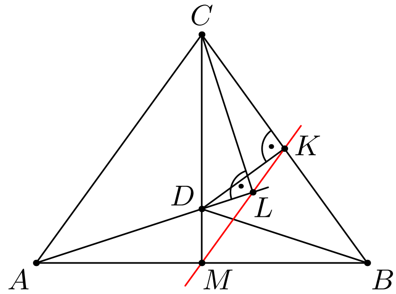

fig. 7

8. Let ABC be a triangle with $AC = BC$ (fig. 8). Point M is the midpoint of AB and point D is the midpoint of CM. Let S be the foot of the perpendicular from M to AD. Prove that the lines BS and CS are perpendicular.

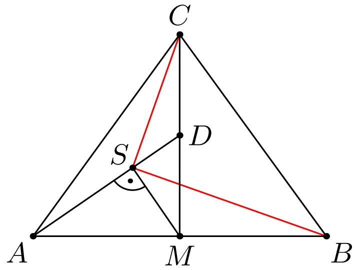

fig. 8

9. Let ABC be a triangle with $AC = BC$ (fig. 9). Point M is the midpoint of AB. Point P lies inside triangle ABC and satisfies $\angle PAB = \angle PBC$. Prove that $\angle APM + \angle BPC = 180°$.

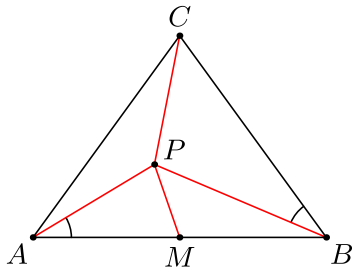

fig. 9

10. In a convex quadrilateral $ABCD$ the following equalities hold (fig. 10)

$$\angle ADB = 2\angle ACB \quad \text{and} \quad \angle BDC = 2\angle BAC\,.$$

Prove that $AD = CD$.

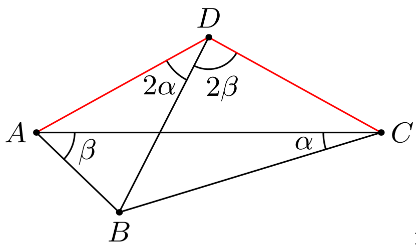

fig. 10

11. Point P lies inside a parallelogram $ABCD$ and satisfies the equalities (fig. 11)

$$\angle DAP = 2\angle PCD \quad \text{and} \quad \angle CBP = 2\angle CDP\,.$$

Prove that $AP = BP = BC$.

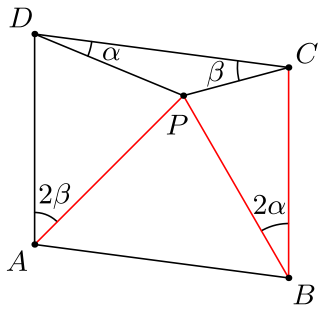

fig. 11

12. Points E and F lie on the sides AB and BC of a square $ABCD$, respectively, such that $BE = BF$ (fig. 12). Point S is the foot of the perpendicular from B to CE. Prove that $\angle DSF = 90°$.

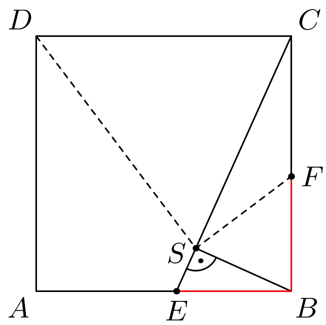

fig. 12

13. Point E lies on the side BC of a square $ABCD$ (fig. 13). Points P and Q are the feet of perpendiculars from E and B to BD and DE, respectively. Prove that the points A, P, Q are collinear.

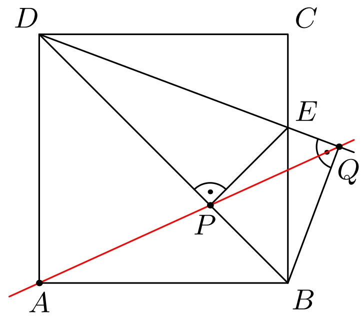

fig. 13

14. Point P lies inside a parallelogram $ABCD$, such that $\angle PAB = \angle PCB$ (fig. 14). Prove that

$$\angle PBA = \angle PDA\,.$$

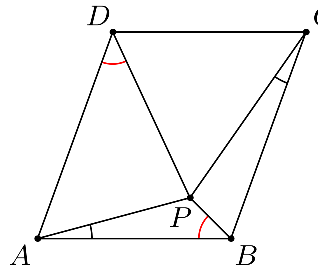

fig. 14

15. Point P lies outside of a parallelogram $ABCD$, such that $\angle PAB = \angle PCB$ (fig. 15). Prove that

$$\angle APB = \angle CPD\,.$$

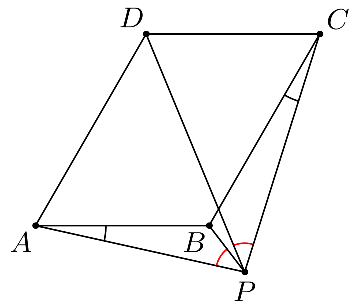

fig. 15

16. Point P lies inside triangle ABC such that the equality $\angle PAC = \angle PBC$ is satisfied (fig. 16). Points K and L are the midpoints of the line segments CP and AB, respectively. Let M be the foot of the perpendicular from P to the angle bisector of the angle ACB. Prove that the points K, L and M are collinear.

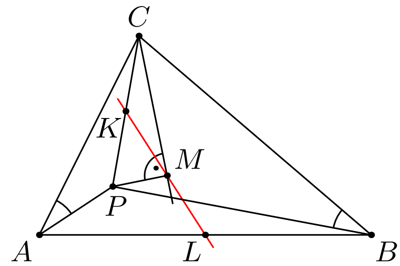

fig. 16

17. A convex quadrliateral $ABCD$ with $AB = BC$ is inscribed in a circle (fig. 17). Point M is the midpoint of AC and K is the foot of the perpendicular from B to CD. Prove that the lines BD and KM are perpendicular.

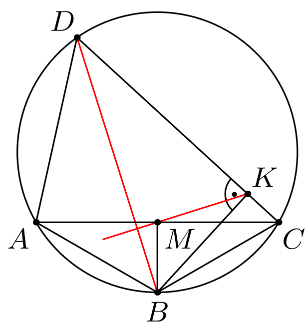

fig. 17

18. From a point P lying outside of a circle with center O two tangents PA and PB are drawn (fig. 18). Point M lies on the line segment AB. The line passing through M and perpendicular to OM intersets the lines AP and BP at K and L, respectively. Prove that $KM = LM$.

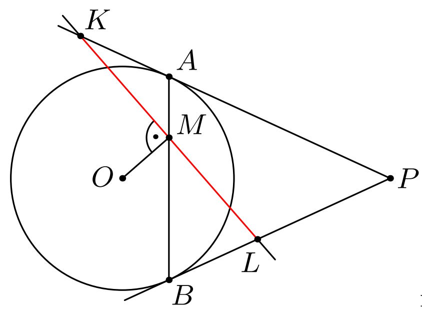

fig. 18

19. An acute-angled triangle ABC is inscribed in a circle ω (fig. 19). The tangents to the circle ω at A and C intersect at F. The perpendicular bisector of the line segment AB intersects the line BC at E. Prove that the lines FE and AB are parallel.

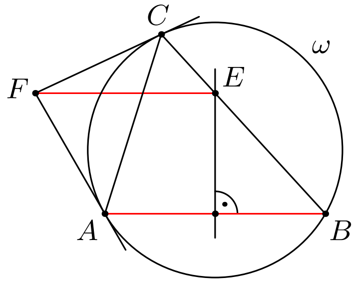

fig. 19

20. Given is a convex quadrilateral $ABCD$ with (fig. 20)

$$\angle BAC = 44^\circ\,, \quad \angle BCA = 17^\circ\,, \quad \angle CAD = \angle ACD = 29^\circ\,.$$

Determine the measure of the angle ABD.

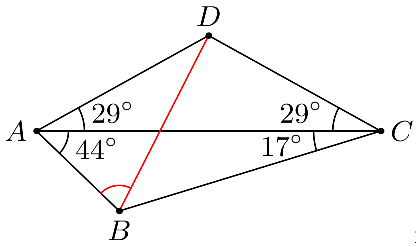

fig. 20

21. The incircle of triangle ABC is tangent to the sides BC and CA at points K and L, respectively (fig. 21). Let I denote the incenter of triangle ABC. The lines AI and KL meet at P. Prove that the lines AP and BP are perpendicular.

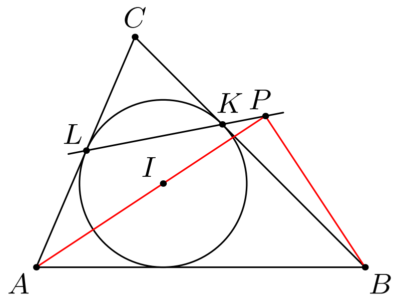

fig. 21

22. The excircle ω of triangle ABC is tangent to the side BC at point D and to the line AC at point E (fig. 22). Let J be the center of ω. Denote by Q the foot of the perpendicular from B to AJ. Prove that the points D, E and Q are collinear.

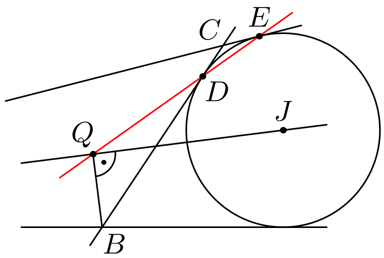

fig. 22

23. Let ABC be an acute-angled triangle and let D be the foot of the perpendicular from A to BC (fig. 23). The incircle of triangle ABC is tangent to the sides BC and CA at points K and L, respectively. Denote by I the incenter of triangle ABC and let E be the reflection of the point D across the line KL. Prove that the lines A, I and E are collinear.

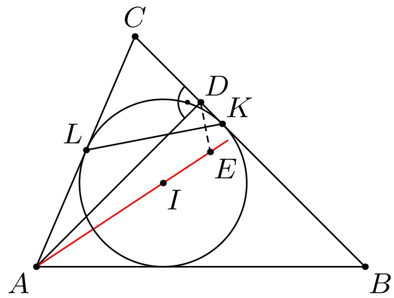

fig. 23

24. In an acute-angled triangle ABC point D is the foot of the perpendicular from the point C to the line AB. (fig. 24). Points K and L are the feet of the perpendiculars from the points A and B, respectively, to the angle-bisector of the angle ACB. Point M is the midpoint of the line segment AB. Prove that the points D, K, L, M are concyclic.

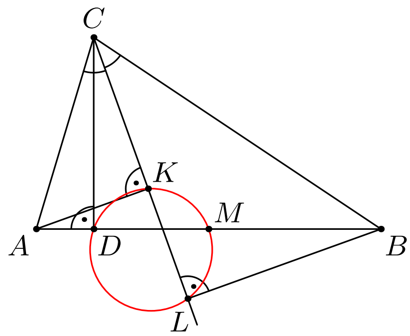

fig. 24

25. Let $ABCD$ be a convex quadrilateral (fig. 25). Prove that there exists an excircle of the quadrilateral $ABCD$ if and only if $AB + BC = AD + DC$.

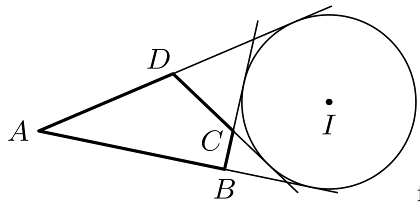

fig. 25

26. Let $ABCD$ be a conave quadrilateral (fig. 26). Prove that there exists an excircle of the quadrilateral $ABCD$ if and only if $AB + BC = AD + DC$.

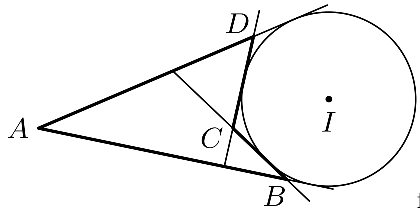

fig. 26

27. Given two circles ω_1 and ω_2 (fig. 27). A transversal common tangent l touches the circles ω_1 and ω_2 at points B and C, respectively. The line l intersects the external common tangents to the circles ω_1 and ω_2 at points A and D. Prove that $AB = CD$.

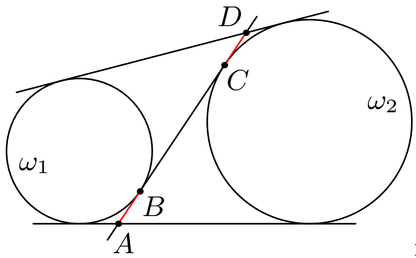

fig. 27

28. An external common tangent to two circles ω_1 and ω_2 touches the circles ω_1 and ω_2 at A and B, respectively (fig. 28). A transversal common tangent to the circles ω_1 and ω_2 intersects their external common tangents at points A and D. Prove that $AB = CD$.

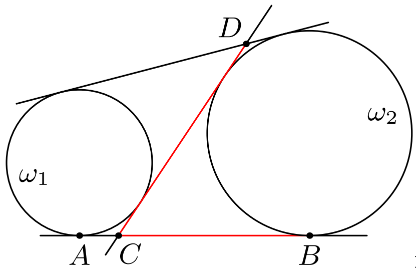

fig. 28

29. Two excircles of triangle ABC are tangent to the sides BC and AC of triangle ABC at points D and E, respectively (fig. 29). Prove that $AE = BD$.

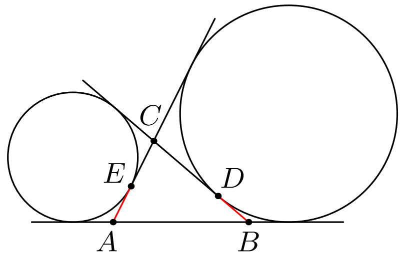

fig. 29

30. Prove that there exists an incircle of quadrilateral $ABCD$ if and only if the incircles of triangles ABD and BCD are tangent (fig. 30).

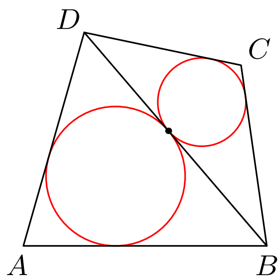

fig. 30

31. Let $ABCD$ be a circumscribed quadrilateral (fig. 31). Point P lies on the side CD. Prove that there exists a common tangent to the incircles of triangles ABP, BCP and ADP.

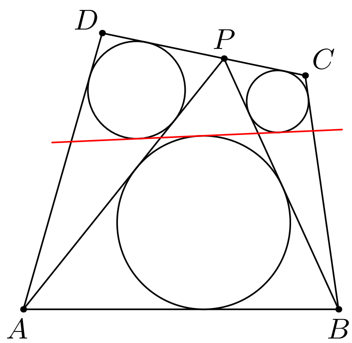

fig. 31

32. Point D lies on side AB of triangle ABC (fig. 32). The incircles of triangles ABC, ADC and BDC are tangent to the line AB at points I, J and K, respectively. Prove that $IJ = DK$.

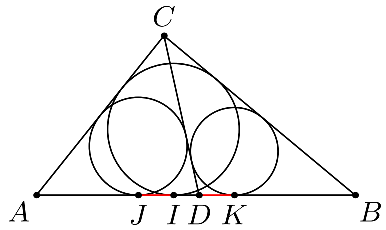

fig. 32

33. A variable point D lies on side AB of a fixed triangle ABC (fig. 33). The external common tangent of the incircles of triangles ADC and BDC (different from the line AB) intersects the line CD at point E. Find the locus of the points E, as D varies on the line segment AB.

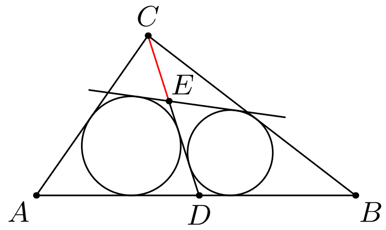

fig. 33

34. Points P and Q lie on sides AB and AD of a convex quadrilateral $ABCD$ (fog. 34). The lines DP and BQ meet at S. Prove that if there exist incircles of quadrilaterals $APSQ$ and $BCDS$, then there exists an incircle of quadrilateral $ABCD$.

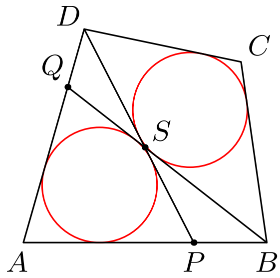

fig. 34

35. Let D be a point lying on side AB of triangle ABC such that $CD = AC$ (fig. 35). The incircle of triangle ABC is tangent to the sides AC and AB at points E and F, respectively. Let I denote the incircle of triangle BCD and let the lines AI and EF meet at P. Prove that $AP = PI$.

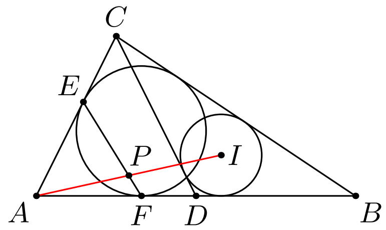

fig. 35

36. A convex quadrilateral $ABCD$ is cut into 9 convex quadrilaterals, as shown on figure 36. Prove that if there exist incircles of the shaded quadrilaterals, then there exists an incircle of quadrilateral $ABCD$.

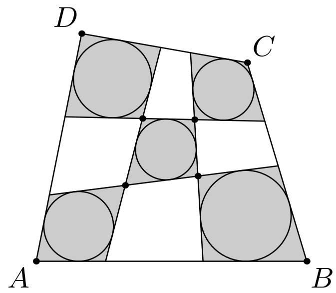

fig. 36

Hints and suggestions

Remark: The suggested attempts to problems are not the only ones. Most of the problems can be solved in several ways. Readers are encouraged to find their own ways to solutions before they read the proposed hints.

1. Note that the points A, B, D, E lie on a common circle, whose center is M. The equality $\angle DME = 60^\circ$ follows from $\angle EMD = 2\angle EBD$.

2. Let H be the intersection of the altitudes of triangle ABC. Observe that the quadrilaterals $AFHE$ and $BFHD$ are cyclic. You will need to note that $\angle CAD = \angle CBE$.

3. Observe that points C, D, F, P are concyclic and that this implies $\angle CPF = 90^\circ$. Next use this to show that A, E, P, F are concyclic. Analogously, show that A, F, Q, E are concyclic and conclude that all the five points lie on a common circle.

4. Let P be the intersection of the line segments AF and BG. Observe that the triangles CGB and CAF are congruent and use it to show that the points A, P, C, G are concyclic, and also B, P, C, F are concyclic. Then determine each of the angles HPA, APB, BPE and conclude that P lies on the line segment EH.

5. Let P be the intersection of the line segments AD and BE. Observe that the triangles ACD and ECB are congruent and use it to show that the quadrilaterals $AECP$ and $BDCP$ are cyclic. Deduce now that the points A, F, B, P are concyclic. Finally, determine the angles CPD, DPB and BPF and conclude that P lies on the line segment CF.

6. Let P be the intersection of the line segments AD and BE. Observe that the triangles ACD and ECB are congruent and use it to show that the quadrilaterals $ACPE$ and $BDPC$ are cyclic. Deduce now that the points A, F, B, P are concyclic. Determine the angles $\angle APC$ and $\angle APF$ and conclude that C lies on the line PF.

7. Observe that the quadrilaterals $AMLC$ and $DLKC$ are cyclic and use it to show that $\angle KLC + \angle ALM = 90^\circ$.

8. Draw a line l parallel to AB through C and let AD and l intersect at E. Show that $MBEC$ is a rectangle and that the points S, M, B, E are concyclic. Conclude that the points B, M, S, C are concyclic.

9. Assume that the circumcircle of triangle BCP intersects the line CM at points C and Q. Show that the points A, P, Q, M are concyclic. (You will need to consider several cases depending on the different possitions of the point Q on the line CM.)

10. Let E be the intersection of the circumcircle of triangle ABC with the line BD. Use the given equalities to show that triangles ADE and CDE are isosceles.
Remark: Observe an extra result: $AD = CD = BD$.

11. Let Q be a point such that $APQD$ is a parallelogram. Observe that $BCQP$ is also a parallelogram. Apply Problem 10 for the quadrilateral $DPCQ$.

12. Assume BS and AD meet at P. Show that triangles ABP and BCE are congruent. Use it next to observe that $PFCD$ is a rectangle. Observe also that the points P, S, C, D are concyclic.

13. Let the lines EP and AB meet at S. Observe that the quadrilaterals $BQEP$, $CEPD$ and $ADPS$ are cyclic.

14. Let Q be a point such that $BCPQ$ is a parallelogram. Observe that the quadrilateral $ADPQ$ is also a parallelogram and that the points A, P, B, Q are concyclic.

15. Let Q be a point such that $ABPQ$ is a parallelogram. Observe that the quadrilateral $PCDQ$ is also a parallelogram and that the points A, Q, P, D are concyclic.

16. Let N be the reflection of the point P with respect to the line CM and denote by Q the point such that $APBQ$ is a parallelogram. Use Problem 15 to show that N lies on the line CQ and then conclude that K, L, M are collinear.

17. Observe that the points B, C, K, M are concyclic. Use it show that $\angle BDC + \angle MKD = 90°$.

18. Observe that the quadrilaterals $AKOM$ and $BOML$ are cyclic and use it to show that the triangle OKL is isosceles.

19. Denote by O the circumcenter of triangle ABC (observe that O lies on the perpendicular bisector of AB). Show that the points A, O, E, C are concyclic by proving that $\angle AOC = \angle AEC$. Observe also that the points A, O, C, F are concyclic. You should consider moreover the case $AC > BC$.

20. Let the line passing through D and perpendicular to the line AC meet the line BC at E. Show that the points A, B, E, D are concyclic.

21. Denote $\alpha = \angle BAI$ and $\beta = \angle ABI$. Observe that the points I, B, P, K are concyclic by computing (in terms of α and β) the angles BIP and BKP. You should also consider the case, where P lies between K and L.

22. Denote $\alpha = \angle CBJ$ and $\beta = \angle BCJ$. Using that the quadrilaterals $BJDQ$ and $JDCE$ are cyclic determine the angles $\angle QDJ$ and $\angle JDE$ and conclude that the points Q, D, E are collinear. Consider also the case, where Q lies between D and E.

23. Let AI and KL meet at P. Use Problem 21 to observe that the points A, B, P, D are concyclic and that the points I, B, P, K are concyclic. Next show that PK is the angle bisector of angle APD and conclude that E lies on the line AI.

24. Let the lines BL and AC meet P. Show that the lines AC and LM are parallel. Next use that the points A, D, K, C are concyclic to show that $\angle MLK = \angle MDK$.

25 and 26. Use the same method as in proof of Theorem 5.

27. Set $x = AB$, $y = BC$, $z = CD$. Let an external common tangent to ω_1 and ω_2 touch ω_1 and ω_2 at P and Q, respectively. Let the second external common tangent to ω_1 and ω_2 touch ω_1 and ω_2 at R and S, respectively. Express PQ and RS in terms of x, y, z and observe that $PQ = RS$.

28. Use Problem 27.

29. Inscribe a circle in triangle ABC and use Problem 27.

30. Let the incircles of triangles ABD and BCD touch the line segment BD at points K and L, respectively. Set $x = BK$, $y = KL$, $z = LD$ and express the sides of quadrilateral $ABCD$ in terms of x, y and z.

31. Let the external common tangent to the incircles of triangles ADP and BCP (different from the line CD) meet AP and BP at R and S, respectively. Use Theorem 3 to show that $AB + RS = AR + BS$ is equivalent to $AB + CD = AD + BC$. In a similar way Problem 36 can be solved.

32. Set $a = BC$, $b = AC$, $c = CD$, $x = AD$, $y = BD$. Use Theorem 4 to find IJ and DK.

33. Set $a = BC$, $b = AC$, $c = CD$, $x = AD$, $y = BD$. Use Theorem 4 and Problem 27 to determine the length of CE.

34. Observe that the given circles are inscribed in quadrilaterals $ABSD$ and $BSDC$. Use Theorem 5 to show that $AB + CD = AD + BC$.

35. Let the line passing through I and parallel EF meet AB at Q. Show that triangle DIQ is isosceles observing that $\angle AFE = \angle IDB$. Next use this to show that $AF = FQ$ (set $a = BC$, $b = AC$, $c = CD$, $x = AD$, $y = BD$ and compute the lengths of AF and FQ).

36. Denote the points of tangency as shown on figure 37. Observe that there exists an incircle of the central quadrilateral if and only if $MN + RQ = ST + PO$. Use this equality to show that $AB + CD = BC + DA$.

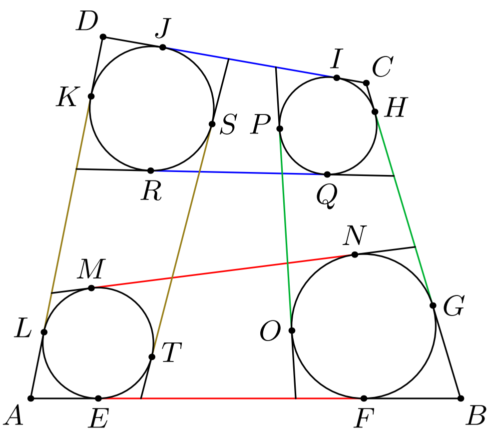

fig. 37