THE POLYNOMIALS

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The First Excursion into the Realm of Polynomials

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INTRODUCTION

A *polynomial of degree* n (where n is a non-negative integer) is an experession $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, where $a_0, a_1, a_2, \ldots, a_n$ are called *coefficients* and are given numbers, $a_n \neq 0$, and x is a non-specified, indeterminate symbol. We write $n = \deg p(x)$.

The symbols $\mathbb{N}[x]$, $\mathbb{Z}[x]$, $\mathbb{Q}[x]$, $\mathbb{R}[x]$ and $\mathbb{C}[x]$ denote the sets of all polynomials with the coefficients taken only from the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ respectively. ($\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ stands for the sets of non-negative intgers, integers, rational numbers, real numbers and complex numbers.) We will often use the notation $\mathbb{F}[x]$ where, if not stated otherwise, the symbol $\mathbb{F}$ stands for anyone of the letters $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$.

Some of the problems presented and solved here may be even solved in other, sometimes more efficient way. The use of the techniques of polynomials was intended here to show the power of this techniques.

Polynomials may be studied in several different contexts and the following three are the most importatnt:

$\star$ In the set $\mathbb{F}[x]$ one may introduce addition and multiplication operations between polynomials, one may define some relations, for example = (equality) between two polynomials, and then study the property of $\mathbb{F}[x]$ without bothering about the nature of the indeterminate x.

$\star$ One can consider replacing the indeterminate x by an arbitrary real numbers, or by rationals, or integers, and then the polynomial $p(x)$ automatically becomes a *polynomial function* defined on $\mathbb{R}, \mathbb{Q}$ or $\mathbb{Z}$. Those x_0 for which $p(x_0) = 0$ are then called *zeros* of the polynomial $p(x)$.

$\star$ One can limit onself to study only those x for which $p(x) = 0$. Then we deal with a *polynomial equation*, and the solutions are called *roots* of the equation. It is obvious that those numbers are the same as the zeros of the corresponding polynomial function.

The operations on polynomials are defined in the natural way: if $p(x) = a_0+a_1x+a_2x^2+\cdots+a_nx^n$ and $q(x) = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m$ then

$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots$ and
$p(x) \cdot q(x) = (a_0b_0) + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \cdots$.

It is then obvious that $\deg(p(x) + q(x)) \leq \max\{\deg p(x), \deg q(x)\}$ while $\deg(p(x) \cdot q(x)) = \deg p(x) + \deg q(x)$.

Two polynomials $p(x)$ and $q(x)$ are equal if $\deg p(x) = \deg q(x)$ and the coefficients for the same power of x are equal.

The polynomial of degree 0, i.e. $p(x) \equiv c$ for som $c \neq 0$, is called the *constant polynomial*. It is convenient to set the degree of the *zero polynomial*, $p(x) \equiv 0$, to -1.

SOME BASIC PROPERTIES

1. Division of polynomials.

In many aspects the polynomials are like integers. Not only one can add, subtract and multiply them but one can also divide polynomials. The polynomial $q(x)$ is said to be a *divisor* of $p(x)$ if there is a polynomial $k(x)$ such that $p(x) = k(x) \cdot q(x)$. The actual division algorithm (long division) is assumed to be known.

Theorem 1. Suppose $p(x), q(x) \in \mathbb{F}[x]$, where $\mathbb{F}$ is $\mathbb{Q}$ or $\mathbb{R}$. Then there exist unique polynomials $k(x), r(x) \in \mathbb{F}[x]$ such that $p(x) = k(x)q(x) + r(x)$, where $\deg r(x) < \deg q(x)$.

The statement above is true even if $\mathbb{F}$ equals $\mathbb{Z}$, provided $q(x)$ is *monic*, which means that the coefficient for the highest power of x equals 1. □

One useful consequence of the last sentence is that if an integer polynomial (i.e. a polynomial with all coefficients in $\mathbb{Z}$) has an integer monic divisor so is the quotient again an integer polynomial.

Example 1. Dividing the polynomial $p(x) \in \mathbb{R}[x]$ by $x-1$ gives the rest 3. Dividing $p(x)$ by $x-2$

gives the rest 4. What is the rest when $p(x)$ is divided by $(x-1)(x-2)$?

Solution. Since $p(x) = k_1(x)(x-1)+3$ then $p(1) = k_1(1)(1-1)+3 = 3$. Similarily, since $p(x) = k_2(x)(x-2)+4$ then $p(2) = k_2(2)(2-2)+4 = 4$.

Now, dividig $p(x)$ by $(x-1)(x-2)$ we get a rest of degree at most 1, $r(x) = ax+b$, so $p(x) = k_3(x)(x-1)(x-2)+(ax+b)$. Using the equalities $p(1) = 3$ and $p(2) = 4$ we get $3 = p(1) = k_3(1)(1-1)(1-2)+(a+b) = a+b$ and $4 = p(2) = k_3(2)(2-1)(2-2)+(2a+b) = 2a+b$. Solving the system of equations $a+b = 3$ and $2a+b = 4$ we find that $a = 1$ and $b = 2$. Hence $r(x) = x+2$. □

Another very important result follows from Theorem 1, namely the so called *Factor Theorem*, or *Bézout Theorem* after the French mathematitien Étienne Bézout from 18th century.

Theorem 2. If $p(x) \in \mathbb{F}[x]$ and α is an arbitrary number then $p(\alpha) = 0$ if and only if the polynomial $(x-\alpha)$ divides $p(x)$. □

Example 2. Let $p(x) \in \mathbb{Z}[x]$ be a monic polynomial. Suppose that for some positive integer k none of the five numbers $p(k)$, $p(k+1)$, $p(k+2)$, $p(k+3)$, $p(k+4)$ is divisible by 5. Prove that $p(x)$ has no rational zeros.

Solution. Suppose in the contrary that $p(x)$ has a rational zero x_0. Then, since $p(x)$ is monic, x_0 is an integer (by theorem 8). Hence, $p(x) = (x-x_0)q(x)$, where $q(x)$ has integer coefficients (a consequence of theorem 1).

Consider now the five numbers $p(k+i) = (k+i-x_0)q(k+i)$, for $i = 0,1,2,3,4$. Since $(k+i-x_0)$, where $i = 0,1,2,3,4$, are five consequtive integers then one of them is divisible by 5. Hence one of the numbers $p(k+i)$ is divisible by 5, which contradicts the assumption in the problem. Thus $p(x)$ has no rational zeros. □

Example 3. (Poland, 1980) The polynomial $p(x) \in \mathbb{Z}[x]$ satisfies $p(a) = p(b) = p(c) = p(d) = 1$ for four distinct integers a, b, c and d. Prove that there is no integer e such that $p(e) = -1$.

Solution. Consider the polynomial $q(x) = p(x) - 1$. Since a, b, c and d are zeros of $q(x)$ then $q(x) = (x-a)(x-b)(x-c)(x-d)h(x)$, where $h(x)$ has integer coefficients (this is a consequence of theorem 1).

Suppose now that $p(e) = -1$ for some integer e. Then it follows that $q(e) = -2$, which is $(e-a)(e-b)(e-c)(e-d)h(e) = -2$. Hence, -2 is written as a product of five integers, at least four of which are distinct (since a, b, c and d are distinct). This is however not possible since at most three of those integers can be distinct, for example $-2 = -1 \cdot 1 \cdot 2$. Thus, such integer e does not exist. □

There is one more basic fact about the polynomials with integer coefficients which may be useful in some situations:

Suppose $p(x) \in \mathbb{Z}[x]$ and $a \neq b$ are two integers. Then $a-b$ divides $p(a) - p(b)$.

(This fact is a simple consequence of the algebraic identity $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 \cdots + ab^{n-2} + b^{n-1}$. Just consider the difference $p(a) - p(b)$ and factor out $a - b$ from each pair of terms of the same degree.)

2. Factorization.

Although not directly needed for solving problems, the folowing theorem plays a central role in the theory of solving polynomial equations.

Theorem 3. (*The Fundamental Theorem of Algebra*) Every polynomial $p(x) \in \mathbb{C}[x]$ of degree ≥ 1 has at least one complex zero. $\square$

The symbol $\mathbb{C}[x]$ means of course the set of all polynomials with complex coefficients. As a special case, it follows that every polynomial $p(x) \in \mathbb{R}[x]$ has at least one complex zero.

The main consequence of Theorem 3 is the following

Theorem 4. Each polynomial $p(x)$ of degree $n \geq 1$ can be written uniquely in the form $p(x) = c(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$, where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are not necessary distinct (complex) numbers and are the only zeros of $p(x)$. $\square$

Another way to express this statment is that each polynomial of degree $n \geq 1$ can be written as $p(x) = c(x-\alpha_1)^{n_1}(x-\alpha_2)^{n_2}\cdots(x-\alpha_k)^{n_k}$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are distinct numbers and the sum of the positive integers $n_1, n_2, \ldots, n_k$ equals n. The integers $n_1, n_2, \ldots, n_k$ are often refered to as *multiplicities* of the zeros $\alpha_1, \alpha_2, \ldots \alpha_k$.

Example 4. Let $p(x) \in \mathbb{C}[x]$. Show that there exists a non-zero polynomial $q(x) \in \mathbb{C}[x]$ such that each exponent of the product $p(x)q(x)$ is a multiple of 2005.

Solution. Let $p(x)$ have the unique factorization $p(x) = b(x-b_1)^{n_1}(x-b_2)^{n_2}\cdots(x-b_k)^{n_k}$, where $b, b_1, b_2, \ldots b_k$ are complex numbers and $n_1, n_2, \ldots n_k$ are multiplicities of the zeros $b_1, b_2, \ldots b_k$.

Now we want to use the algebraic identity $A^n - B^n = (A-B)(A^{n-1} + A^{n-2}B + A^{n-3}B^2 + \cdots + A^2B^{n-3} + AB^{n-2} + B^{n-1})$, i.e. $\dfrac{A^n - B^n}{A - B} = A^{n-1} + A^{n-2}B + A^{n-3}B^2 + \cdots + A^2B^{n-3} + AB^{n-2} + B^{n-1}$ for $A \neq B$. Thus

$\dfrac{x^{2005} - b_i^{2005}}{x - b_i} = x^{2004} + x^{2003}b_i + x^{2002}b_i^2 + \cdots + x^2b_i^{2002} + xb_i^{2003} + b_i^{2004}$, which is then a polynomial.

This suggests that we take

$q(x) = x^{2005}\Big(\dfrac{x^{2005} - b_1^{2005}}{x - b_1}\Big)^{n_1}\Big(\dfrac{x^{2005} - b_2^{2005}}{x - b_2}\Big)^{n_2}\cdots\Big(\dfrac{x^{2005} - b_k^{2005}}{x - b_k}\Big)^{n_k}$. Hece we get

$p(x)q(x) = b(x-b_1)^{n_1} \cdots (x-b_k)^{n_k} x^{2005} \Big(\dfrac{x^{2005}-b_1^{2005}}{x-b_1}\Big)^{n_1} \cdots \Big(\dfrac{x^{2005}-b_k^{2005}}{x-b_k}\Big)^{n_k} =$
$bx^{2005}\big(x^{2005}-b_1^{2005}\big)^{n_1} \cdots \big(x^{2005}-b_k^{2005}\big)^{n_k}$, which is a polynomial with all exponents divisible by 2005. □

Next two very useful theorms can be easily derived from Theorem 3.

Theorem 5. Suppose $\deg p(x) \le n$ and $p(x)$ has at least $n+1$ zeros. Then $p(x) \equiv 0$. □

Theorem 6. Suppose that two polynomial functions $p(x), q(x)$ of degree at most n has the same value for at least $n+1$ distinct x. Then $p(x) = q(x)$ for all x, i.e. $p(x) \equiv q(x)$. □

Example 5. (USA, 1975) Find all polynomials $p(x)$ such that $p(x) = \dfrac{1}{2}\big(p(x+1)+p(x-1)\big)$ and $p(0) = 0$.

Solution. We show by induction that $p(n) = np(1)$ for all non-negative integers n. Since the statement is true for $n = 0$ and $n = 1$ so suppose it is true for some $n = k-1$ and $n = k$, where $k \ge 1$.

For $n = k+1$ we have then $p(k+1) = 2p(k) - p(k-1) = 2kp(1) - (k-1)p(1) = (k+1)p(1)$. Hence, by the induction principle $p(n) = np(1)$ for all non-negative integers n.

Consider the polynomial $q(x) = p(x) - x \cdot p(1)$. For all $n = 0, 1, 2, 3, \ldots$. Then $q(n) = p(n) - n \cdot p(1) = 0$, meaning that $q(x)$ has infinitely many zeros. Thus $q(x) \equiv 0$, i.e. $p(x) = p(1)x$ for all x. □

Example 6. (Putnam, 1971) Determine all polynomials $p(x)$ such that $p(x^2+1) = \big(p(x)\big)^2 + 1$ and $p(0) = 0$.

Solution. Let's start by checking some values for x. We have $p(0) = 0,\ p(1) = p(0^2+1) = \big(p(0)\big)^2 + 1 = 1,\ p(2) = p(1^2+1) = \big(p(1)\big)^2 + 1 = 2,\ p(5) = p(2^2+1) = \big(p(2)\big)^2 + 1 = 5,$ $p(26) = p(5^2+1) = \big(p(5)\big)^2 + 1 = 26$ and so on. This should suggest that maybe $p(x) = x$ for all x.

To show this, let define a sequence $\{x_n\}$ of integers by $x_0 = 0$ and $x_n = x_{n-1}^2 + 1$ for $n > 0$. Then suppose that for some $n > 0$, $p(x_{n-1}) = x_{n-1}$. Hence $p(x_n) = p(x_{n-1}^2+1) = \big(p(x_{n-1})\big)^2 + 1 = x_{n-1}^2 + 1 = x_n$. By the induction, $p(x_n) = x_n$ for all $n \in \mathbb{N}$.

Since then $p(x) = x$ for an infinite number of x then, by theorem 6, $p(x) = x$ for all x. □

FEW MORE USEFUL THEOREMS

3. Viète's identities.

The French 16th century mathematician François Viète was the first to show the relation between the coefficients of a polynomial equation and its roots. Considering the second degree equation $ax^2 + bx + c = 0$ he proved that the roots x_1, x_2 satisfy $x_1 + x_2 = -\dfrac{b}{a}$ and $x_1 x_2 = \dfrac{c}{a}$. For the third degree equation $ax^3 + bx^2 + cx + d = 0$ and its roots x_1, x_2, x_3 the corresponding relations are $x_1 + x_2 + x_3 = -\dfrac{b}{a}$, $x_1x_2 + x_1x_3 + x_2x_3 = \dfrac{c}{a}$ and $x_1x_2x_3 = -\dfrac{d}{a}$.

Generally we have the following theorem:

Theorem 7. Suppose $x_1, x_2, \ldots, x_n$ are the roots of the polynomial equation $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$. Then

$$\sum_{i=1}^{n} x_i = -\frac{a_{n-1}}{a_n},$$

$$\sum_{1\le i<j\le n} x_i x_j = \frac{a_{n-2}}{a_n},$$

$$\sum_{1\le i<j<k\le n} x_i x_j x_k = -\frac{a_{n-3}}{a_n},$$

and so on, untill

$$x_1 x_2 \cdots x_n = (-1)^n \frac{a_0}{a_n}.$$

□

Example 7. Find $a \in \mathbb{R}$ such that the sum of squares of roots of the equation $x^3 - ax^2 + (a + 1)x + 11 = 0$ is minimal.

Solution. Let x_1, x_2, x_3 be the roots of $x^3 - ax^2 + (a+1)x + 11 = 0$. Then $x_1^2 + x_2^2 + x_3^2 = (x_1 + x_2 + x_3)^2 - 2(x_1x_2 + x_1x_3 + x_2x_3) =$ (using now Viète's identities) $= a^2 - 2(a+1) = (a-1)^2 - 3$. The minimum value of the last expression is obviously -3 and is attained when $a = 1$. □

Example 8. Suppose that a, b, c are real numbers such that $a + b + c = 0$. Prove that $a^3 + b^3 + c^3 = 3abc$.

Solution. Consider the polynomial equation $x^3 + \alpha x^2 + \beta x + \gamma = 0$ whose roots are the numbers a, b and c (it is obviously the equation $(x - a)(x - b)(x - c) = 0$).

According to the Viète's identities $a + b + c = -\alpha$, $ab + ac + bc = \beta$ and $abc = -\gamma$. Thus, since $\alpha = 0$, the equation reduces to $x^3 + \beta x + \gamma = 0$.

Since a, b and c are the roots of the last equation then $a^3 + \beta a + \gamma = 0$, $b^3 + \beta b + \gamma = 0$ and $c^3 + \beta c + \gamma = 0$. Adding all three equalities yields $(a^3 + b^3 + c^3) + \beta(a + b + c) + 3\gamma = 0$. Finally, since the second parenthesis equals 0 then we have $(a^3 + b^3 + c^3) + 3\gamma = 0$, which means $(a^3 + b^3 + c^3) = 3abc$. □

4. Zeros of some polynomials.

There are a few "good-to-know" theorems useful in solving some special polynomial equations. We present here the most applicable of them. The first one gives us a method to find the eventual rational zeros of a polynomial with integer coefficients. Given such a polynomial we can always create a finite list of possible rational zeros and then we may just check them one by one.

Theorem 8. Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ and assume that $x = \dfrac{s}{t}$ is a rational root of the equation $p(x) = 0$, where the greatest common divisor of s and t is 1. Then s is a divisor of a_0 and t is a divisor of a_n. □

Example 9. (Poland, 1973) Let $p(x) = ax^3 + bx^2 + cx + d \in \mathbb{Z}[x]$ with all zeros being real numbers. Suppose that ad is an odd number while bc is an even number. Show that at least one of zeros of $p(x)$ is irrational.

Solution. Suppose that all three roots x_1, x_2, x_3 of the equation $p(x) = 0$ are rational. Then the numbers $y_k = ax_k$, for $k = 1, 2, 3,$ are rational roots of the equation $y^3 + by^2 + acy + a^2 d = 0$. Since all rational rots of this equation are in fact integers (all coefficients are integers and the coefficient at the highest exponent of x is 1) then each one of the y_k is a divisor of $a^2 d$, thus is an odd number.

Now, since $-b = y_1 + y_2 + y_3$ and $ac = y_1 y_2 + y_1 y_3 + y_2 y_3$ then both numbers b and ac are odd. Hence b and c are odd, which contradicts the fact that the product bc is even. This proves that not all three roots x_1, x_2, x_3 can be rational. □

The next theorem tells us that the eventual complex zeros of a real polynomial never stands alone, but come in conjugate pairs. Knowing one such a zero $z = \alpha + \beta i$ we know in fact even one more, namely $\overline{z} = \alpha - \beta i$ and we may divide the given polynomial by (i.e. factor out) the real second degree polynomial $\big(x - (\alpha + \beta i)\big)\big(x - (\alpha - \beta i)\big) = x^2 - 2\alpha x + (\alpha^2 + \beta^2)$.

Theorem 9. Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{R}[x]$. If $z = \alpha + \beta i$ is a (complex) root of the equation $p(x) = 0$ the $\overline{z} = \alpha - \beta i$ is also a root of this equation. □

Example 10. Suppose $p(x) \in \mathbb{R}[x]$ is of degree 5 and $p^2(x)$ cannot be written in the form $p^2(x) = q_1^2(x) + q_2^2(x)$, where $q_1(x), q_2(x)$ are real polynomials with different degrees. Show that all zeros of $p(x)$ are real.

Solution. By theorem 9, complex zeros of $p(x)$ come in conjugate pairs. Hence $p(x)$ may have two or four complex zeros. At least one zero is then real. If not all of zeros are real then we have to possibility to consider: (1) $p(x) = a(x - a_1)(x - a_2)(x - a_3)\big(x - (b + ci)\big)\big(x - (b - ci)\big)$ or (2) $p(x) = a(x - a_1)\big(x - (b_1 + c_1 i)\big)\big(x - (b_1 - c_1 i)\big)\big(x - (b_2 + c_2 i)\big)\big(x - (b_2 - c_2 i)\big)$, where $a, a_1, a_2, a_3, b, b_1, b_2, c, c_1, c_2$ are all real numbers.

Case (1): Since $\big(x - (b + ci)\big)\big(x - (b - ci)\big) = (x - b)^2 + c^2$ then we have $p^2(x) = a^2(x - a_1)^2(x - a_2)^2(x - a_3)^2\big((x - b)^2 + c^2\big)^2$.

In the next step we may have use of an algebraic identity saying that a product of a sum of two squares with another sum of two squares is again a sum of two squares, namely $(A^2+B^2)(C^2+D^2)=(AD+BC)^2+(AC-BD)^2$.

In case when A,B,C,D are polynomials with $\deg A > \deg B$ and $\deg C > \deg D$ then we easily check that $\deg\,(AC-BD) > \deg\,(AD+BC)$.

Thus, $\big((x-b)^2+c^2\big)^2$ can be written as a sum $p_1^2(x)+p_2^2(x)$ of squares of two polynomials of different degree. Hence $p^2(x)=a^2(x-a_1)^2(x-a_2)^2(x-a_3)^2\big(p_1^2(x)+p_2^2(x)\big)=\big(a(x-a_1)(x-a_2)(x-a_3)p_1(x)\big)^2+\big(a(x-a_1)(x-a_2)(x-a_3)p_2(x)\big)^2$, where both terms are real polynomials of different degrees. This contradicts the assumption in the problem.

Case (2): We have again $p^2(x)=a^2(x-a_1)^2\big((x-b_1)^2+c_1^2\big)^2\big((x-b_2)^2+c_2^2\big)^2$. Using the same algebraic identity several times we find that $\big((x-b_1)^2+c_1^2\big)^2\big((x-b_2)^2+c_2^2\big)^2=p_1^2(x)+p_2^2(x)$ where $p_1^2(x),\ p_2^2(x)$ are real polynomials of different degrees.

Hence, $p^2(x)=a^2(x-a_1)^2\big(p_1^2(x)+p_2^2(x)\big)=\big(a(x-a_1)p_1(x)\big)^2+\big(a(x-a_1)p_2(x)\big)^2$, with both terms of different degree. Again, a contradiction.

These two contradictions prove that all zeros of $p(x)$ are real. □

Remark: The statement of Example 10 is obviously valid for all non-constant real polynomials, not only those of degree 5. The proof is almost identical. In fact, the converse statement is also true: If all zeros of $p(x)\in\mathbb{R}[x]$ are real then $p^2(x)$ cannot be expressed as $q_1^2(x)+q_2^2(x)$, where $q_1(x),q_2(x)$ are real polynomials with different degrees.

Theorem 10. The number $x=a$ is a root of the polynomial equation $p(x)=0$ with the multiplicity $k\ge 1$ if and only if $x=a$ is a root of the equation $p'(x)=0$ with the multiplicity $k-1$, where $p'(x)$ means the derivative of $p(x)$. Consequently, $x=a$ is a root of the polynomial equation $p(x)=0$ with the multiplicity $k\ge 1$ if and only if $x=a$ is a root of the equations $p(x)=0$, $p'(x)=0$, $p''(x)=0$, $p^{(3)}(x)=0,\ldots,\ p^{(k-1)}(x)=0$, i.e. the root of the first $k-1$ derivatives of $p(x)$. □

Example 11. (Poland, 1977) Prove that for $n=1,2,3,\ldots$ the polynomial $Q_n(x)=1+\dfrac{x}{1!}+\dfrac{x^2}{2!}+\cdots+\dfrac{x^n}{n!}$ has no zeros of multiplicity greater than 1.

Solution. Suppose, in the contrary, that α is a zero of $Q_n(x)$ with multiplicity $k>1$. Then $(x-\alpha)^k$ divides $Q_n(x)$ and $(x-\alpha)^{k-1}$ divides the derivative $Q_n'(x)=Q_{n-1}(x)$. Thus $(x-\alpha)$ divides $P(x)=Q_n(x)-Q_{n-1}(x)=\dfrac{x^n}{n!}$. Hence, by Factor Theorem, $P(\alpha)=0$, which yields $\dfrac{\alpha^n}{n!}=0$, i.e. $\alpha=0$. However, $Q_n(0)\neq 0$, proving that no such α exists. □

Example 12. Let $p(x)\in\mathbb{R}[x]$ and suppose $(x-1)^k$ divides $p(x^m)$ that for some positive integers m and k. Show that $(x^m-1)^k$ divides $p(x^m)$.

Solution. Since $(x-1)^k$ divides $p(x^m)$ then $p(x^m)=(x-1)^k p_1(x)$ for some polynomial $p_1(x)$.

Derivating this expression gives $p'(x^m) = k(x-1)^{k-1}p_1(x) + (x-1)^k p_1'(x) = (x-1)^{k-1}p_2(x)$ for some polynomial $p_2(x)$. We may contiune derivating another $k-2$ times and finally $p^{(k-1)}(x^m) = (x-1)^{k-1}p_k(x)$ for some polynomial $p_k(x)$.

Letting now $x = 1$ into those experssions we find that $p(1) = p'(1) = p''(1) = \cdots = p^{(k-1)}(1) = 0$. Hence, by the theorem above,, $(x-1)^k$ divides $p(x)$. Finally, substituting x by x^m gives that $(x^m - 1)^k$ divides $p(x^m)$. □

5. Greatest common divisor, GCD.

A polynomial $h(x)$ is called a *greatest common divisor* of $p(x)$ and $q(x)$, we write $h(x) = \text{GCD}\big(p(x), q(x)\big)$, if (1) $h(x)$ divides $p(x)$ and $q(x)$, and (2) if $k(x)$ is any other polynomial that divides $p(x)$ and $q(x)$ then $k(x)$ divides $h(x)$. It follows that $\text{GCD}\big(p(x), q(x)\big)$ is unique up to a constant multiple.

The procedure of finding $\text{GCD}\big(p(x), q(x)\big)$ is much the same as the euclidean algorithm for finding the $\text{GCD}(m, n)$ for $m, n \in \mathbb{N}$. The outline of this procedure (the euclidean algorithm for polynomials) is the following:

Given two polynomials $p(x)$ and $q(x)$ of degree $m \geq n \geq 0$ respectively. Dividing $p(x)$ by $q(x)$ yields

$$p(x) = k_1(x)q(x) + r_1(x),$$

where $k_1(x)$ and $r_1(x)$ are the quotients and the rest polynomials respectively. Of course $\deg r_1(x) < \deg q(x)$.

If $r_1(x)$ is not the zero polynomial then, in the next step, we divide $q(x)$ by $r_1(x)$:

$$q(x) = k_2(x)r_1(x) + r_2(x),$$

where $\deg r_2(x) < \deg r_1(x)$.

In the following steps we continue dividing the previous divisor by the previous rest polynomial, provided it is not the zero polynomial. Since the degree of the rest polynomial is always decreasing we will, after some steps, end up with the rest $r_{k+1}(x) \equiv 0$. Then $\text{GCD}\big(p(x), q(x)\big) = r_k(x)$, i.e. $\text{GCD}\big(p(x), q(x)\big)$ equals the last non-zero rest polynomial in performing the euclidean algorithm. □

Example 13. Find the $\text{GCD}\big(x^8 - 1, x^5 - 1\big)$.

Solution. Following the euclidean algorithm we find that

$$\begin{aligned}
x^8 - 1 &= (x^5 - 1)x^3 + (x^3 - 1),\\
x^5 - 1 &= (x^3 - 1)x^2 + (x^2 - 1),\\
x^3 - 1 &= (x^2 - 1)x + (x - 1),\\
x^2 - 1 &= (x - 1)(x + 1).
\end{aligned}$$

Thus, $\text{GCD}\big(x^8 - 1, x^5 - 1\big) = x - 1$, the last non-zero rest polynomial. □

Theorem 11. Suppose $h(x) = \text{GCD}\big(p(x), q(x)\big)$. Then there exist two polynomials $s(x), t(x)$

such that $h(x) = s(x)p(x) + t(x)q(x)$. Those polynomials may be obtained by reversing the steps of the euclidean algorithm. $\square$

Example 14. Does there exist polynomials $p(x), q(x)$ such that $(x^8-1)p(x)+(x^5-1)q(x) = x-1$?

Solution. In the previous exercise we found that $\text{GCD}\big(x^8-1, x^5-1\big) = x-1$. Hence, by theorem 11, such polynomials exist. Following the euclidean algorithm in the reverse direction we find now that
$x - 1 = (x^3-1) - (x^2-1)x = (x^3-1) - \big((x^5-1) - (x^3-1)x^2\big)x = (x^3-1) - (x^5-1)x + (x^3-1)x^3 = (x^3-1)(x^3+1) - (x^5-1)x = \big((x^8-1) - (x^5-1)x^3\big)(x^3+1) - (x^5-1)x = (x^8-1)(x^3+1) - (x^5-1)x^3(x^3+1) - (x^5-1)x = (x^8-1)(x^3+1) - (x^5-1)(x^6+x^3+x)$.
Hence $p(x) = x^3 + 1$ and $q(x) = x^6 + x^3 + x$. $\square$

Theorem 12. The number $x = a$ is a common zero of the polynomials $p(x)$ and $q(x)$ if and only if $x = a$ is a zero of $\text{GCD}\big(p(x), q(x)\big)$. $\square$

Example 15. Solve the equation $x^6 - 2x^5 - 12x^4 + 24x^3 + 20x^2 - 24x - 16 = 0$, knowing that it has a root with multiplicity greater than 1.

Solution. Let $p(x) = x^6 - 2x^5 - 12x^4 + 24x^3 + 20x^2 - 24x - 16$. Being optimistic and seeing a polynomial with integer coefficients we may hope it has rational (in this case integer) zeros. This however is not the case which may be confirmed after some, rather tedious examination of possible cases (theorem 8). What remains is another, unfortunately not less tedious approach.

Since $p(x) = 0$ has a root α which is at least double then α is a common root of $p(x) = 0$ and $p'(x) = 0$. This common root is, according to theorem 12, a root of the equation $\text{GCD}\big(p(x), p'(x)\big) = 0$. Hence, what we need is to find $q(x) = \text{GCD}\big(p(x), p'(x)\big)$.

This task is time consuming and after some serious calculations (euclidean algorithm) we will eventually find that $q(x) = x^2-2x-2$. Thus, both roots of the equation $q(x) = 0$, i.e. $\alpha_1 = 1+\sqrt{3}$ and $\alpha_2 = 1 - \sqrt{3}$ are double roots of $p(x) = 0$ and we may divide $p(x)$ by $\big(q(x)\big)^2 = (x^2 - 2x - 2)^2 = x^4 - 4x^3 + 8x + 4$. We will get $p(x) = (x^2 - 2x - 2)^2(x^2 + 2x - 4)$.

The remaining two roots of $p(x) = 0$ are the roots of the equation $x^2 + 2x - 4 = 0$, i.e. $\alpha_3 = -1 + \sqrt{5}$ and $\alpha_4 = -1 - \sqrt{5}$.

Hence the given equation has two double roots, $\alpha_1 = 1 + \sqrt{3}$ and $\alpha_2 = 1 - \sqrt{3}$, and two single roots, $\alpha_3 = -1 + \sqrt{5}$ and $\alpha_4 = -1 - \sqrt{5}$. $\square$

6. Reciprocal polynomials.

The polynomial $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ is called *reciprocal*, if $a_k = a_{n-k}$, for $k = 0, 1, 2, \ldots, n$. For example $x^{2005} + 1$ and $4x^6 - 17x^5 + 11x^3 - 17x + 4$ are reciprocal.

Theorem 13. Any reciprocal polynomial $p(x)$ of degree $2n$ can be written in the form $p(x) =$

$x^n q(z)$, where $z = x + \frac{1}{x}$, and $q(z)$ is a polynomial in z of degree n.

The suggested substitution implies that $x^2 + \frac{1}{x^2} = z^2 - 2,\ x^3 + \frac{1}{x^3} = z^3 - 3z,\ x^4 + \frac{1}{x^4} = z^4 - 4z^2 + 2,\ x^5 + \frac{1}{x^5} = z^5 - 5z^3 + 5z,$ and so on. □

It is not difficult to verify the following properties of the reciprocal polynomials:

(1) Every polynomial $p(x)$ of degree n and with $a_0 \neq 0$ is reciprocal if and only if $x^n p(\frac{1}{x}) = p(x)$.

(2) Every reciprocal polynomial $p(x)$ of odd degree is divisible by $x + 1$ and the quotient is a reciprocal polynomial of even degree.

(3) If α is a zero of a reciprocal polynomial then $\frac{1}{\alpha}$ is also a zero of this polynomial.

Example 16. Solve the equation $x^8 + 4x^6 - 10x^4 + 4x^2 + 1 = 0$.

Solution. Since $x = 0$ is not a root of this reciprocal equation we may divide it by x^4, getting $x^4 + \frac{1}{x^4} + 4x^2 + \frac{4}{4x^2} - 10 = 0$. The substitution $z = x + \frac{1}{x}$ reduces this equation to $z^4 = 16$, which has four roots: $z_1 = 2$, $z_2 = -2$, $z_3 = 2i$ and $z_4 = -2i$.

From the equality $z = x + \frac{1}{x}$ we get $x = \frac{z \pm \sqrt{z^2 - 4}}{2}$ and substituting now the four values of z we get the eight roots of the given equation: $x_1 = x_2 = 1$, $x_3 = x_4 = -1$, $x_5 = (1 + \sqrt{2})i$, $x_6 = (1 - \sqrt{2})i$, $x_7 = (-1 + \sqrt{2})i$ and $x_8 = (-1 - \sqrt{2})i$. □

COLLECTION OF PROBLEMS

Here follows the first set of problems for training in polynomials. To each problem there is given a hint, but it is not necessary to follow it in order to find the solution. There, as almost always, are many different ways to approach a mathematical problem. The suggested complete solutions are given later in the text.

1. Find a polynomial $p(x)$ such that $p(x)$ is divisible by $x^2 + 1$ and $p(x) + 1$ is divisible by $x^3 + x^2 + 1$.

(Hint: Note that GCD $(x^2 + 1, x^3 + x^2 + 1) = 1$.)

2. Suppose a, b, c are real numbers such that $A = a + b + c$, $B = ab + bc + ac$ and $C = abc$ are positive numbers. Show that $a, b, c > 0$.

(Hint: The form of A, B and C suggests an introduction of a polynomial of degree 3.)

3. Find the numbers a and b such that such that $(x-1)^2$ is a divisor of $ax^4+bx^3+x-2005$.

(Hint: Multiple roots.)

4. Suppose that a, b, c are real numbers such that $a+b+c=0$. Prove that

(1) $2a^4+2b^4+2c^4=(a^2+b^2+c^2)^2$,

(2) $\dfrac{a^5+b^5+c^5}{5}=\dfrac{a^2+b^2+c^2}{2}\cdot\dfrac{a^3+b^3+c^5}{3}$,

(3) $\dfrac{a^7+b^7+c^7}{7}=\dfrac{a^2+b^2+c^2}{2}\cdot\dfrac{a^5+b^5+c^5}{5}$.

(Hint: Use the technique and the result of Example 8.)

5. (Sweden, 1989) Prove that the polynomial $p(x)=x^6-x^5+x^4-x^3+x^2-x+\dfrac{3}{4}$ has no real zeros.

(Hint: Show that $p(x)$ never changes sign.)

6. (Poland, 1994) Find all polynomials $p(x)$ of degree at most 5 such that $(x-1)^3$ is a divisor of $p(x)+1$ and $(x+1)^3$ is a divisor of $p(x)-1$.

(Hint: Derivate both expressions $p(x)+1=(x-1)^3q_1(x)$ and $p(x)-1=(x+1)^3q_2(x)$.)

7. (Putnam, 1977) Consider all lines which meet the graph $y=2x^4+7x^3+3x-5$ in four distinct points, say (x_i, y_i), for $i=1,2,3,4$. Show that the sum $x_1+x_2+x_3+x_4$ is independent of the line and find its value.

(Hint: Make use of the Viète's identities.)

8. Find all real polynomials $p(x)$ satisfying $p(x^2)+p(x)p(x+1)=0$ for all $x\in\mathbb{R}$.

(Hint: Show that if x_0 is a zero of the polynomial $p(x)$ then even x_0^2 is a zero of this polynomial.)

9. Solve the following equation:
$4x^{11}+4x^{10}-21x^9-21x^8+17x^7+17x^6+17x^5+17x^4-21x^3-21x^2+4x+4=0.$

(Hint: Use the fact that on the left-hand side is a reciprocal polynomial of odd degree.)

10. (BalticWay, 1991) Let $p(x)\in\mathbb{Z}[x]$. Prove that if $p(-n)<p(n)<n$ for some integer n, then $p(-n)<-n$.

(Hint: Use the fact (metioned in the beginning) that for $p(x)\in\mathbb{Z}[x]$ and two integers $a\neq b$ the number $a-b$ divides $p(a)-p(b)$.)

11. (USA, 1975) A polynomial $p(x)$ of degree n satisfies the conditions $p(k)=\dfrac{k}{k+1}$, for $k=0,1,2,\ldots,n$. Find $p(n+1)$.

(Hint: It may be interesting to study the polynomial $q(x)=(x+1)p(x)-x$.)

12. (Russia, 1992) It is allowed to transform the polynomial $p(x) = ax^2 + bx + c$ into either $p_1(x) = x^2 p\big(1 + \frac{1}{x}\big)$ or to $p_2(x) = (x-1)^2 p\big(\frac{1}{x-1}\big)$. Applying this procedure several times, is it possible to obtain $x^2 + 10x + 9$ when starting with $x^2 + 4x + 3$?

(Hint: Consider the discriminant: the discriminat for $p(x) = \alpha x^2 + \beta x + \gamma$ is the number $\beta^2 - 4\alpha\gamma$.)

13. (Singapore, 1978) The polynomial $p(x) \in \mathbb{R}[x]$ of degree n satisfies for two real numbers a and b, where $a < b$, the following inequalities:

$p(a) < 0$ and $(-1)^k p^{(k)}(a) \leq 0$ for $k = 1, 2, \ldots, n$, and

$p(b) > 0$ and $(-1)^k p^{(k)}(b) \geq 0$ for $k = 1, 2, \ldots, n$.

Prove that all real zeros of $p(x)$ are in the interval (a, b).

(Hint: Consider the polynomials $q(x) = p(a - x)$ and $r(x) = p(b + x)$. What can be said about the coefficients of these polynomials?)

14. (Australia, 1990) Let $p(x) \in \mathbb{Z}[x]$ and suppose $a, b \in \mathbb{Z}$, $a \neq b$, satisfy $p(a)p(b) = -(a-b)^2$. Prove that $p(a) + p(b) = 0$.

(Hint: Consider the numbers $A = \dfrac{p(a)}{a-b}$ and $B = \dfrac{-p(b)}{a-b}$.)

15. The polynomial $p(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + 1$ with nonnegative coefficients $a_1, a_2, \ldots, a_{n-1}$ has n real zeros. Prove that $p(2) \geq 3^n$.

(Hint: Viète's formulas and the AM-GM Inequality.)

16. (Bulgaria, 1976) Find all polynomials $p(x) \in \mathbb{R}[x]$ such that $p(x^2 - 2x) = \big(p(x-2)\big)^2$.

(Hint: Putting $y = x - 1$ and $q(y) = p(y - 1)$ transforms the given equality into $q(y^2) = \big(q(y)\big)^2$.)

17. (Sovjet, 1991) Given $2n$ distinct numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$, an $n \times n$ table is filled as follows: into the cell in the i-th row and j-th column is written the number $a_i + b_j$. Prove that if the product of the numbers in each coulmn is the same, then the product of the numbers in each row is the same.

(Hint: Consider the polynomial $p(x) = (x + a_1)(x + a_2)\cdots(x + a_n) - (x - b_1)(x - b_2)\cdots(x - b_n)$.)

18. (Iran, 1992) Let $p(x) \in \mathbb{Q}[x]$ and suppose that the real number α satisfy $\alpha^3 - \alpha = \big(p(\alpha)\big)^3 - p(\alpha) = 33^{1992}$. Prove that for each integer $n \geq 1$ the equality $\big(p^n(\alpha)\big)^3 - pn(\alpha) = 33^{1992}$ holds, where $p^n(x) = p\Big(p\big(p(\ldots p(x)\ldots)\big)\Big)$, n applications of $p(x)$.

(Hint: Show first that the equation $x^3 - x - 33^{1992} = 0$ has only one real root.)

19. (Belarus, 1993) Find all polynomials $p(x) \in \mathbb{R}[x]$ such that $1 + p(x) = \dfrac{p(x-1) + p(x+1)}{2}$

for all real x.

(Hint: Find that x^2 is one solution then consider the polynomial $q(x) = p(x) - x^2$. What can be said about $q(x) - q(x-1)$?)

20. (IMO, 1976) Let $p_1(x) = x^2 - 2$ and $p_{k+1}(x) = p_1\big(p_k(x)\big)$, for $k = 1, 2, 3, \ldots$. Show that, for any positive integer n, the roots of the equation $p_n(x) = x$ are real and distinct.

(Hint: Try the substitution $x = 2\cos t$, which transform the interval $[0, \pi]$ onto $[-2, 2]$.)

21. (IMO, 1993) Let $p(x) = x^n + 5x^{n-1} + 3$ where $n > 1$ is an integer. Prove that $p(x)$ cannot be expressed as a product of two non-constant polynomials with integer coefficients.

(Hint: Suppose that $p(x) = q_1(x) \cdot q_2(x)$,where $\deg q_1(x), \deg q_2(x) > 0$, both factors have integer coefficients, $q_1(0) = \pm 1$, $q_2(0) = \mp 3$ and $q_1(x) = -x^k + a_{k-1}x^{k-1} + \cdots + a_1 x \pm 1$. Study the zeros of $q_1(x)$.)

22. (IMO, 1973) Let a and b be real numbers for which the equation $x^4 + ax^3 + bx^2 + ax + 1 = 0$ has at least one real root. Find the least possible value of $a^2 + b^2$.

(Hint: Since the equation is reciprocal we may start by dividing the equation by taking $y = x + \dfrac{1}{x}$, which reduces the equation to $y^2 + ay + b - 2 = 0$. When will this equation have a real root y such that the equation $y = x + \dfrac{1}{x}$ has a real root x?)

PROOFS OF THE THEOREMS

Here we give proofs of most of the theorems presented in an earlier part of this text.

Theorem 1. Suppose $p(x), q(x) \in \mathbb{F}[x]$, where $\mathbb{F}$ is $\mathbb{Q}$ or $\mathbb{R}$. Then there exist unique polynomials $k(x), r(x) \in \mathbb{F}[x]$ such that $p(x) = k(x)q(x) + r(x)$, where $\deg r(x) < \deg q(x)$.

The statement above is true even if $\mathbb{F}$ equals $\mathbb{Z}$, provided $q(x)$ is *monic*, which means that the coefficient for the highest power of x equals 1.

Proof. The existence of $k(x)$ and $r(x)$ follows from the division algorithm. The only thing which remains to prove is the uniqueness of $k(x)$ and $r(x)$. Suppose then that $p(x) = k_1(x)q(x) + r_1(x)$ and $p(x) = k_2(x)q(x) + r_2(x)$, where $\deg r_1(x), \deg r_1(x), < \deg q(x)$.

Subtracting the second equation from the first yields $0 = \big(k_1(x) - k_2(x)\big)q(x) + \big(r_1(x) - r_2(x)\big)$, where the 0 on the left hand side means the zero polynomial.

If $k_1(x) \neq k_2(x)$ then $\deg \big(k_1(x) - k_2(x)\big)q(x) > \deg \big(r_1(x) - r_2(x)\big)$, which obviously gives a contradiction: the right hand side cannot be a zero polynomial. Hence $k_1(x) = k_2(x)$ and this implies that $r_1(x) = r_2(x)$ as well. $\square$

Theorem 2. If $p(x) \in \mathbb{F}[x]$ and α is an arbitrary number then $p(\alpha) = 0$ if and only if the polynomial $(x - \alpha)$ divides $p(x)$.

Proof. ($\Rightarrow$) Since $\deg(x - \alpha) = 1$, then dividing $p(x)$ by $(x - \alpha)$ gives the rest of degree at most 0. Hence $p(x) = (x - \alpha)k(x) + r$, where r is a constant. Letting in $x = \alpha$ we get $p(\alpha) = (\alpha - \alpha)k(\alpha) + r$, which implies that $r = 0$. Thus $(x - \alpha)$ divides $p(x)$.

($\Leftarrow$) Since $p(x) = (x - \alpha)k(x)$ then, letting $x = \alpha$, we have $p(\alpha) = (\alpha - \alpha)k(x) = 0$. □

Theorem 3. (*The Fundamental Theorem of Algebra*) Every polynomial $p(x) \in \mathbb{C}[x]$ of degree ≥ 1 has at least one complex zero.

Comment. This theorem was proved by Carl Friedrich Gauss in 1799 in his doctoral dissertation and the proof is just to advanced for this text. □

Theorem 4. Each polynomial $p(x)$ of degree $n \geq 1$ can be written uniquely in the form $p(x) = c(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$, where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are not necessary distinct (complex) numbers and are the only zeros of $p(x)$.

Proof. The proof is by induction. The case of a polynomial of degree $n = 1$ is obvious. Suppose then that the statment is valid for all polynomials of degree $n \geq 1$ and let $p(x)$ be a polynomial of degree $n + 1$.

By the theorem 3, $p(x)$ has a complex zero α_1 and thus, by theorem 2, $p(x) = c(x - \alpha_1)q(x)$, where $\deg q(x) = n$. According to the assumption $q(x)$ can be factorized in the first degree factors times a constant, and then the statment follows. □

Theorem 5. Suppose $\deg p(x) \leq n$ and $p(x)$ has at least $n + 1$ zeros. Then $p(x) \equiv 0$.

Proof. If $p(x)$ is not a zero polynomial and has degree n then, by theorem 4, $p(x) = c(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$ where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the only zeros of $p(x)$. Hence $p(x)$ cannot have more than n zeros, unless it is a zero polynomial. □

Theorem 6. Suppose that two polynomial functions $p(x), q(x)$ of degree at most n has the same value for at least $n + 1$ distinct x. Then $p(x) = q(x)$ for all x, i.e. $p(x) \equiv q(x)$.

Proof. Consider the polynomial $h(x) = p(x) - q(x)$ and suppose $p(x_k) = q(x_k)$ for $k = 1, 2, \ldots, n + 1$. Then $h(x_k) = 0$ for $k = 1, 2, \ldots, n + 1$ and the theorem 6 implies that $h(x) \equiv 0$, which means that $p(x) \equiv q(x)$. □

Theorem 7. Suppose $x_1, x_2, \ldots, x_n$ are the roots of the polynomial equation $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$. Then

$$\sum_{i=1}^{n} x_i = -\frac{a_{n-1}}{a_n},$$

$$\sum_{1\le i<j\le n} x_i x_j = \frac{a_{n-2}}{a_n},$$

$$\sum_{1\le i<j<k\le n} x_i x_j x_k = -\frac{a_{n-3}}{a_n},$$

and so on, until

$$x_1 x_2 \cdots x_n = (-1)^n \frac{a_0}{a_n}.$$

Proof. Instead for a complete proof we only show the validity of the statement for a polynomial equation of degree 3, $a_3x^3 + a_2x^2 + a_1x + a_0 = 0$. In general case the proof follows the same idea.

According to the theorem 3, $a_3x^3 + a_2x^2 + a_1x + a_0 = a_3(x - x_1)(x - x_2)(x - x_3)$, while $a_3(x-x_1)(x-x_2)(x-x_3) = a_3x^3 - a_3(x_1+x_2+x_3)x^2 + a_3(x_1x_2+x_2x_3+x_3x_1)x - a_3x_1x_2x_3$. Thus we have two equal polynomials: $a_3x^3 + a_2x^2 + a_1x + a_0$ and $a_3x^3 - a_3(x_1 + x_2 + x_3)x^2 + a_3(x_1x_2 + x_2x_3 + x_3x_1)x - a_3x_1x_2x_3$.

Identifying the coefficients iat the same power of x we get have $a_2 = -a_3(x_1 + x_2 + x_3)$, $a_1 = a_3(x_1x_2+x_2x_3+x_3x_1)$ and $a_0 = a_3x_1x_2x_3$. Thus $x_1+x_2+x_3 = -\frac{a_2}{a_3}$, $x_1x_2+x_2x_3+x_3x_1 = \frac{a_1}{a_3}$ and $x_1x_2x_3 = -\frac{a_0}{a_3}$. □

Theorem 8. Let $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in \mathbb{Z}[x]$ and assume that $x = \frac{s}{t}$ is a rational root of the equation $p(x) = 0$, where the greatest common divisor of s and t is 1. Then s is a divisor of a_0 and t is a divisor of a_n.

Proof. Since $p(\frac{s}{t}) = a_n\big(\frac{s}{t}\big)^n + a_{n-1}\big(\frac{s}{t}\big)^{n-1} + \cdots + a_1\big(\frac{s}{t}\big) + a_0 = 0$, then, multiplying both sides by t^n we get $(\star)$ $a_ns^n + a_{n-1}s^{n-1}t + a_{n-2}s^{n-2}t^2 + \cdots + a_2s^2t^{n-2} + a_1st^{n-1} + a_0t^n = 0$.

Moving the last term to the right hand side and factoring out s yields $s\big(a_ns^{n-1} + a_{n-1}s^{n-2}t + a_{n-2}s^{n-3}t^2 + \cdots + a_2st^{n-2} + a_1t^{n-1}\big) = -a_0t^n$. Note that all numbers in this expression are integers. Since s divides the left hand side then s is a divisor of a_0t^n as well. We know however that $\text{GCD}(s,t) = 1$. Thus s is a divisor of a_0.

Moving instead the first term of $(\star)$ to the right hand side and factoring out t will give $t\big(a_{n-1}s^{n-1} + a_{n-2}s^{n-2}t + \cdots + a_2s^2t^{n-3} + a_1st^{n-2} + a_0t^{n-1}\big) = -a_ns^n$. Since t divides the left hand side then t is a divisor of a_ns^n as well. Again, because $\text{GCD}(s,t) = 1$, then t is a divisor of a_n. The proof is complete. □

Theorem 9. Let $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in \mathbb{R}[x]$. If $z = \alpha + \beta i$ is a (complex) root of the equation $p(x) = 0$ the $\overline{z} = \alpha - \beta i$ is also a root of this equation.

Proof. In the proof we make use of the following properties of complex numbers: (1) $\overline{z} + \overline{w} =$

$\overline{z+w}$, (2) $\overline{z}\cdot\overline{w}=\overline{zw}$, (3) $\overline{z}^n=\overline{z^n}$ and (4) $\overline{z}=z$ if and only if $z\in\mathbb{R}$.

Suppose that $z=\alpha+\beta i$ is a root of $p(x)=0$ where the coefficients of $p(x)$ are real. Then we have

$$p(\overline{z})=a_n\overline{z}^n+a_{n-1}\overline{z}^{n-1}+\dots+a_2\overline{z}^2+a_1\overline{z}+a_0\overset{(3)}{=}a_n\overline{z^n}+a_{n-1}\overline{z^{n-1}}+\dots+a_2\overline{z^2}+a_1\overline{z}+a_0\overset{(4)}{=}$$
$$\overline{a_n}\overline{z^n}+\overline{a_{n-1}}\overline{z^{n-1}}+\dots+\overline{a_2}\overline{z^2}+\overline{a_1}\cdot\overline{z}+\overline{a_0}\overset{(2)}{=}\overline{a_nz^n}+\overline{a_{n-1}z^{n-1}}+\dots+\overline{a_2z^2}+\overline{a_1z}+\overline{a_0}\overset{(1)}{=}$$
$$\overline{a_nz^n+a_{n-1}z^{n-1}+\dots+a_2z^2+a_1x+a_0}=\overline{p(z)}=\overline{0}=0.$$

Thus $\overline{z}$ is a root of $p(x)=0$ as well. □

Theorem 10. The number $x=a$ is a root of the polynomial equation $p(x)=0$ with the multiplicity $k\geq 1$ if and only if $x=a$ is a root of the equation $p'(x)=0$ with the multiplicity $k-1$, where $p'(x)$ means the derivative of $p(x)$. Consequently, $x=a$ is a root of the polynomial equation $p(x)=0$ with the multiplicity $k\geq 1$ if and only if $x=a$ is a root of the equations $p(x)=0$, $p'(x)=0$, $p''(x)=0$, $p^{(3)}(x)=0,\ \dots,\ p^{(k-1)}(x)=0$, i.e. the root of the first $k-1$ derivatives of $p(x)$.

Proof. $(\Rightarrow)$ Suppose that $x=a$ is a root of the polynomial equation $p(x)=0$ with the multiplicity $k\geq 1$. Then $p(x)=(x-a)^kq(x)$ and $p'(x)=k(x-a)^{k-1}q(x)+(x-a)^kq'(x)=(x-a)^{k-1}\big(kq(x)+(x-a)q'(x)\big)$. Thus $x=a$ is a root of the equation $p'(x)=0$ with the multiplicity $k-1$.

$(\Leftarrow)$ Given a polynomial $p(x)$, suppose that $x=a$ is a root of the equation $p'(x)=0$ with the multiplicity $k-1$, i.e. $p'(x)=(x-a)^{k-1}q(x)$. Thus, we have to find an antiderivative of $p'(x)$ being a product $(x-a)^kq_1(x)$.

Writing $q(x)=b_mx^m+b_{m-1}x^{m-1}+\dots+b_1x+b_0$, we wish to evaluate $\int p'(x)dx=\int(x-a)^{k-1}\big(b_mx^m+b_{m-1}x^{m-1}+\dots+b_1x+b_0\big)dx=\sum_{i=0}^{m}b_i\int(x-a)^{k-1}x^idx$. It will be sufficient to show that for each $i=0,1,2,\dots,m$ the polynomial $\int(x-a)^{k-1}x^idx$, with the additive constant $c=0$, is divisible by $(x-a)^k$.

This may be done by induction. For $i=0$ we have $\int(x-a)^{k-1}dx=\frac{1}{k}(x-a)^kx$ (remember, as the additive constant c we take 0).

Suppose the statment is true for $i=0,1,\dots,t-1$. Then, integrating by parts, we get $\int(x-a)^{k-1}x^tdx=\frac{1}{k}(x-a)^kx^t-t\int(x-a)^{k-1}x^{t-1}dx$. The last integral is, by the inductive assumption, divisible by $(x-a)^k$. Thus the left hand side is divisible by $(x-a)^k$ as well and we are done. □

Theorem 11. Suppose $h(x)=\mathrm{GCD}\big(p(x),q(x)\big)$. Then there exist two polynomials $s(x),t(x)$ such that $h(x)=s(x)p(x)+t(x)q(x)$.

Comment. Those two polynomials may be obtained by reversing the steps of the euclidean algorithm, as in example ???. $\square$

Theorem 12. The number $x = a$ is a common zero of the polynomials $p(x)$ and $q(x)$ if and only if $x = a$ is a zero of $h(x) = \text{GCD}\big(p(x), q(x)\big)$. $\square$

Proof. According to theorem 11 there exist two polynomials $s(x), t(x)$ such that $h(x) = s(x)p(x) + t(x)q(x)$. If $p(a) = q(a) = 0$ then it is obvious that $h(a) = 0$ as well.

On the other hand, if $h(a) = 0$ then $(x - a)$ divides $h(x)$, and, since $h(x) = \text{GCD}\big(p(x), q(x)\big)$, then $(x - a)$ divides $p(x)$ and $q(x)$ as well. Thus $p(a) = q(a) = 0$. $\square$

Theorem 13. Any reciprocal polynomial $p(x)$ of degree $2n$ can be written in the form $p(x) = x^n q(z)$, where $z = x + \dfrac{1}{x}$, and $q(z)$ is a polynomial i z of degree n.

Proof. We can write $p(x) = a_0x^{2n} + a_1x^{2n-1} + a_2x^{2n-2} + \cdots + a_2x^2 + a_1x + a_0 = x^n\Big(a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + \dfrac{a_2}{x^{n-2}} + \dfrac{a_1}{x^{n-1}} + \dfrac{a_0}{x^n}\Big) = x^n\Big(a_0\big(x^n + \dfrac{1}{x^n}\big) + a_1\big(x^{n-1} + \dfrac{1}{x^{n-1}}\big) + a_2\big(x^{n-2} + \dfrac{1}{x^{n-2}}\big) + \cdots + a_n\Big)$.

Now we have to express $x^k + \dfrac{1}{x^k}$ by using $z = x + \dfrac{1}{x}$:

$$x^2 + \frac{1}{x^2} = \Big(x + \frac{1}{x}\Big)^2 - 2 = z^2 - 2,$$

$$x^3 + \frac{1}{x^3} = \Big(x + \frac{1}{x}\Big)^3 - 3x - \frac{3}{x} = z^3 - 3z,$$

$$x^4 + \frac{1}{x^4} = \Big(x + \frac{1}{x}\Big)^4 - 4x^2 - 6 - \frac{4}{x^2} = z^4 - 4\Big(x^2 + \frac{1}{x^2}\Big) - 6 = z^4 - 4z^2 + 2,$$

$$x^5 + \frac{1}{x^5} = \Big(x + \frac{1}{x}\Big)^5 - 5x^3 - 10x - \frac{10}{x} - \frac{5}{x^3} = z^5 - 5\Big(x^3 + \frac{1}{x^3}\Big) - 10\Big(x + \frac{1}{x}\Big) = z^5 - 5z^3 + 5z,$$

and so on. $\square$

SOLUTIONS TO THE PROBLEMS OF THE COLLECTION

1. We have $p(x) = (x^2 + 1)q_1(x)$ and $p(x) + 1 = (x^3 + x^2 + 1)q_2(x)$ for some polynomials $q_1(x)$, $q_2(x)$. Thus $(x^2 + 1)q_1(x) = p(x) = (x^3 + x^2 + 1)q_2(x) - 1$, i.e. $(x^3 + x^2 + 1)q_2(x) - (x^2 + 1)q_1(x) = 1$.

Note also that $\text{GCD}\,(x^2 + 1, x^3 + x^2 + 1) = 1$, so we can find $q_1(x)$, $q_2(x)$ by working the euclidean algorithm backwards. According to the euclidean algorithm we find that $x^3 + x^2 + 1 = (x + 1)(x^2 + 1) - x$ and $x^2 + 1 = x \cdot x + 1$. Thus $1 = (x^2 + 1) + x \cdot (-x) = (x^2 + 1) + x\big((x^3 +$

$x^2+1)-(x+1)(x^2+1)\big) = x(x^3+x^2+1)-(x^2+1)(x^2+x-1)$. Hence $q_1(x) = x^2+x-1$ and $q_2(x) = x$.

Finally, $p(x) = (x^2+1)q_1(x) = (x^2+1)(x^2+x-1) = x^4+x^3+x-1$.

2. The numbers a, b, c are (by the Viète's identities) the roots of the equation $x^3-Ax^2+Bx-C = 0$. The left-hand side is negative for negative x and equals C for $x = 0$. Hence all three roots of this equation must be positive.

3. Since $x = 1$ is a root of the equation $p(x) = ax^4+bx^3+x-2005 = 0$ with the multiplicity 2 then $p(1) = p'(1) = 0$. This gives two conditions on a and b: $a+b+1-2005 = 0$ and $4a+3b+1 = 0$. From these conditions it is easy to derive $a = -6011$ and $b = 8015$.

4. (1) In the solution of Example 8 it was shown that a, b, c are the roots of $x^3+\beta x+\gamma = 0$, where $\beta = ab+ac+bc$ and $\gamma = -abc$. Thus we have three identities ($\star$) $a^3+\beta a+\gamma = 0$, $b^3+\beta b+\gamma = 0$ and $c^3+\beta c+\gamma = 0$.

Multiplying the expressions ($\star$) by a, b and c respectively and then adding together we get $a^4+b^4+c^4+\beta(a^2+b^2+c^2)+\gamma(a+b+c) = 0$.

Now, since $a+b+c = 0$ and $\beta = ab+ac+bc = \frac{1}{2}\Big((a+b+c)^2-(a^2+b^2+c^2)\Big) = -\frac{a^2+b^2+c^2}{2}$, then $a^4+b^4+c^4-\frac{(a^2+b^2+c^2)^2}{2} = 0$ and we are done.

(2) Multiplying the expressions ($\star$) by a^2, b^2 and c^2 respectively and then adding together we get $a^5+b^5+c^5+\beta(a^3+b^3+c^3)+\gamma(a^2+b^2+c^2) = 0$.

We already know that $\beta = -\frac{a^2+b^2+c^2}{2}$ and, according to Example 8, $\gamma = -abc = -\frac{a^3+b^3+c^3}{3}$. This yields $a^5+b^5+c^5 = -\beta(a^3+b^3+c^3)-\gamma(a^2+b^2+c^2) = \frac{a^2+b^2+c^2}{2}(a^3+b^3+c^3)+\frac{a^3+b^3+c^3}{3}(a^2+b^2+c^2) = \frac{5}{6}(a^2+b^2+c^2)(a^3+b^3+c^3)$, from which the desired equality follows immediately.

(3) The same method as above starting by multiplying the expressions ($\star$) by a^4, b^4 and c^4 respectively and then adding together and using (2).

5. Obviously $p(x) > 0$ for $x \le 0$. Thus there are no non-positive real zeros. Moreover $p(x) = x^5(x-1)+x^3(x-1)+x(x-1)+\frac{3}{4} > 0$ for $x \ge 1$. Hence the possible real zeros satisfy $0 < x < 1$.

Writing now $p(x) = x^5(x-1)+x^3(x-1)+x(x-1)+\frac{3}{4} = x(x-1)(x^4+x^2+1)+\frac{3}{4} = -x(1-x)(x^4+x^2+1)+\frac{3}{4}$ we find for $0 < x < 1$ that $0 < x(1-x) = x-x^2 = -(x-\frac{1}{2})^2+\frac{1}{4} \le \frac{1}{4}$, while $1 < x^4+x^2+1 < 3$. Hence $p(x) = -x(1-x)(x^4+x^2+1)+\frac{3}{4} > -\frac{3}{4}+\frac{3}{4} = 0$.

Thus $p(x) > 0$ for all $x \in R$, and consequently has no real zeros.

6. We need to find all polynomials $p(x)$ of degree at most 5 such that $p(x)+1 = (x-1)^3q_1(x)$ and $p(x)-1 = (x+1)^3q_2(x)$ for some polynomials $q_1(x)$ and $q_2(x)$.

Derivating both expressions yields $p'(x) = 3(x-1)^2 q_1(x) + (x-1)^3 q_1'(x)$ and $p'(x) = 3(x+1)^2 q_2(x) + (x+1)^3 q_2'(x)$. Thus the polynomial $p'(x)$ is divisible by $(x-1)^2$ and $(x+1)^2$. Since $(x-1)^2$ and $(x+1)^2$ have no common factors and $p'(x)$ has degree at most 4 then $p'(x) = a(x-1)^2(x+1)^2 = a(x^2-1)^2$ for a real constant a.

From this we find that $p(x) = \int a(x^2-1)^2 dx = \frac{a}{5}x^5 - \frac{2a}{3}x^3 + ax + c$, for some constant c.

Letting $x = 1$ and $x = -1$ into the expressions in the beginning of the solution we find that $p(1) = -1$ and $p(-1) = 1$. On the other hand, putting $x = 1$ and $x = -1$ into the last expression for $p(x)$ yields $p(1) = \frac{a}{5} - \frac{2a}{3} + a + c = \frac{8}{15}a + c$ and $p(-1) = -\frac{a}{5} + \frac{2a}{3} - a + c = -\frac{8}{15}a + c$.

Hence we have two equations: $\frac{8}{15}a + c = -1$ and $-\frac{8}{15}a + c = 1$. The only solution to this equational system is $(a, c) = (-\frac{15}{8}, 0)$ and the final answer is $p(x) = -\frac{3}{8}x^5 + \frac{10}{8}x^3 - \frac{15}{8}x$.

7. Consider the line $y = kx + m$ intersecting the curve in four distinct points (x_i, y_i), for $i = 1, 2, 3, 4$. Then x_1, x_2, x_3, x_4 are the roots of the equation $2x^4 + 7x^3 + 3x - 5 = kx + m$, or equivalently, the roots of $x^4 + \frac{7}{2}x^3 + \frac{3-m}{2}x - \frac{-5-b}{2} = 0$.

According to the Viète's identities $x_1 + x_2 + x_3 + x_4 = -\frac{7}{2}$

8. If the polynomial $p(x)$ is constant, $p(x) \equiv c$, then, inserting it into the equation gives $c = 0$ or $c = -1$. Both polynomials are apparently solutions to the equation. So let us now assume that $p(x)$ is not constant.

Suppose x_0 is a zero of $p(x)$. Putting x_0 into the equation yields $p(x_0^2) + p(x_0)p(x_0+1) = 0$, i.e. $p(x_0^2) = 0$. Thus x_0^2 is a zero of $p(x)$ as well. This argument can be repeated and, by induction, one shows that $x_0^{2^n}$ are zeros of $p(x)$ for all $n \in \mathbb{N}$. Since the polynomial $p(x)$ has only a finite number of zeros then x_0 can only equals 0, 1 or -1.

Letting now $x_0 - 1$ into the equation yields $p\big((x_0-1)^2\big) + p(x_0-1)p(x_0) = 0$, i.e. $p\big((x_0-1)^2\big) = 0$. This means that $(x_0-1)^2$ is again a zero of $p(x)$. In the view of the above discussion $(x_0-1)^2$ equals 0, 1 or -1. Hence, x_0 can only equals 0 or 1 and then $p(x) = cx^n(x-1)^m$ for some $c \in \mathbb{R}$ and $m, n \in \mathbb{N}$. If $c = 0$, we get the zero polynomial $p(x) \equiv 0$ already considered. Suppose then that $c \neq 0$

Inserting this expression into the equation gives $cx^{2n}(x^2-1)^m + cx^n(x-1)^m \cdot c(x+1)^n x^m = 0$, which reduces to $x^{n-m}(x+1)^{m-n} + c = 0$ for all x. Then apparently $m = n$ and $c = -1$. Hence $p(x) = -x^n(x-1)^n$ for all $n \in \mathbb{N}$.

One must now only check that these functions really satisfy the given equation. Thus the answer is $p(x) \equiv 0$ or $p(x) \equiv -1$ or $p(x) = -x^n(x-1)^n$ for all $n \in \mathbb{N}$.

9. Since any reciprocal polynomial $p(x)$ of odd degree is divisible by $x + 1$ the $x_1 = -1$ is certainly a root of the equation. Dividing the polynomial on the left-hand side by $x + 1$ yields $(x+1)(4x^{10} - 21x^8 + 17x^6 + 17x^4 - 21x^2 + 4) = 0$.

Since $x = 0$ is not a solution we can divide now $4x^{10} - 21x^8 + 17x^6 + 17x^4 - 21x^2 + 4 = 0$ by x^5, getting $4x^5 + \frac{4}{x^5} - 21x^3 - \frac{21}{x^3} + 17x + \frac{17}{x} = 0$

Using now the substitution $z = x + \frac{1}{x}$, we transform the equation into $4z^5 - 41z^3 + 100z = 0$, i.e. $z(4z^4-41z^2+100)=0$. One solution is $z_1=0$, the other four are the roots of the bi-quadratic $4z^4-41z^2+100=0$. These are $z_2=2$, $z_3=-2$, $z_4=\frac{5}{2}$ and $z_5=-\frac{5}{2}$.

From the equality $z=x+\frac{1}{x}$ we get $x=\frac{z\pm\sqrt{z^2-4}}{2}$ and substituting now the five values of z we get the remaining ten roots of the given equation: $x_2=i$, $x_3=-i$, $x_4=x_5=-1$, $x_6=x_7=1$, $x_8=2$, $x_9=\frac{1}{2}$, $x_{10}=-2$ and $x_{11}=-\frac{1}{2}$.

10. We use the fact (metioned in the beginning) that for $p(x)\in\mathbb{Z}[x]$ and two integers $a\neq b$ the number $a-b$ divides $p(a)-p(b)$.

Since $2n=n-(-n)$ divides $p(n)-p(-n)\neq 0$ then $2n\le p(n)-p(-n)$. Hence $p(-n)\le p(n)-2n<n-2n=-n$.

11. Since for $k=0,1,2,\dots,n$ we have $(k+1)p(k)-k=0$ so it may be interesting to study the polynomial $q(x)=(x+1)p(x)-x$.

Since the numbers $0,1,2,\dots,n$ are zeros of $q(x)$ and $\deg q(x)=n+1$ then $q(x)=cx(x-1)(x-2)\cdots(x-n)$ for some constant c.

From this we find that $(x+1)p(x)=x+q(x)=x+cx(x-1)(x-2)\cdots(x-n)$ and, by taking $x=-1$, we get $0=-1+c(-1)^{n+1}(n+1)!$, i.e. $c=\frac{(-1)^{n+1}}{(n+1)!}$. Finally we find that $p(x)=\frac{x+cx(x-1)(x-2)\cdots(x-n)}{x+1}=\frac{x+\frac{(-1)^{n+1}}{(n+1)!}x(x-1)(x-2)\cdots(x-n)}{x+1}$, and taking $x=n+1$ yields $p(n+1)=\frac{n+1+(-1)^{n+1}}{n+2}$.

12. The polynomial $p(x)=ax^2+bx+c$ may be transformed to $p_1(x)=x^2p\big(1+\frac{1}{x}\big)=(a+b+c)x^2+(2a+b)x+a$ or to $p_2(x)=(x-1)^2p\big(\frac{1}{x-1}\big)=cx^2+(-b-2c)x+(a+b+c)$.

In the problems dealing with a repeating procedure it is a good strategy to look after an invariant, a propriety that doesn't change when applying the procedure. Having here a second degree polynomial it is a good start to try to find out if the discriminant for this polynomial is an invariant.

The discriminant for $p(x)$ is b^2-4ac. The discriminant for $p_1(x)$ is $(2a+b)^2-4a(a+b+c)=b^2-4ac$ and the discriminant for $p_2(x)$ is $(-b-2c)^2-4c(a+b+c)=b^2-4ac$.

Hence the discriminant is an invariant for the given transformation. However the discriminant for $x^2+10x+9$ is $100-36=64$, while the discriminant for x^2+4x+3 is $16-12=4$. Thus the answer is: no!

13. Let $q(x)=p(a-x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_x+a_0$. Then $a_0=q(0)=p(a)<0$, $a_1=q'(0)=-p'(a)\le 0$, $a_2=\frac{q''(0)}{2!}=\frac{(-1)^2p''(a)}{2!}\le 0$, $\dots$, $a_n=\frac{q^{(n)}(0)}{n!}=\frac{(-1)^np^{(n)}(a)}{n!}\le 0$.

Since all coefficients of $q(x)$ are ≤ 0 and $a_0<0$ then $q(x)<0$ for all $x\ge 0$. This means that $p(a-x)<0$ for all $x\ge 0$, i.e. $p(x)<0$ for all $x\le a$. Thus, no real zeros of $p(x)$ are in the interval $(-\infty,a)$.

Let now $r(x)=p(b+x)=b_nx^n+b_{n-1}x^{n-1}+\cdots+b_x+b_0$. Then again, $b_0=r(0)=p(b)>0$,

$b_1 = r'(0) = -p'(b) \geq 0,\ b_2 = \dfrac{r''(0)}{2!} = \dfrac{p''(b)}{2!} \geq 0,\ \ldots,\ b_n = \dfrac{r^{(n)}(0)}{n!} = \dfrac{p^{(n)}(b)}{n!} \geq 0.$

Since all coefficients of $r(x)$ are ≥ 0 and $b_0 > 0$ then $r(x) > 0$ for all $x \geq 0$. This means that $p(b+x) > 0$ for all $x \geq 0$, i.e. $p(x) > 0$ for all $x \geq -b$. Thus, no real zeros of $p(x)$ are in the interval (b, ∞).

Consequently, since neither a nor b are zeros of $p(x)$, then every real zero (if such exists) of $p(x)$ must be in the interval (a, b).

14. Let $A = \dfrac{p(a)}{a-b}$ and $B = \dfrac{-p(b)}{a-b}$. Since $p(a) - p(b)$ is divisible by $a - b$ then $p(a) - p(b) = k(a-b)$ for some integer k and $A + B = k$. Moreover $A \cdot B = \dfrac{p(a)}{a-b} \cdot \dfrac{-p(b)}{a-b} = \dfrac{-p(a)p(b)}{(a-b)^2} = 1.$ Thus the rational numbers A and B are the roots of the equation $x + \dfrac{1}{x} = k$, i.e. $x^2 - kx + 1 = 0.$

Since the rational roots of this equation are integers, then either $A = B = 1$ or $A = B = -1$. In any case, $A - B = 0$, which implies $A = \dfrac{p(a) + p(b)}{a-b} = 0$, and consequently $p(a) + p(b) = 0$.

15. Let $x_1, x_2, \ldots, x_n$ be the roots of the polynomial. All those numbers are negative, because all the coefficients are non-negative. Since $p(x) = (x - x_1)(x - x_2) \cdots (x - x_n)$, where $x_i < 0$ $(i = 1, 2, \ldots, n)$ then, taking $b_i = -x_i > 0$ we have $p(x) = (x + b_1)(x + b_2) \cdots (x + b_n)$.

By the Viète's identities we have $x_1 x_2 \cdots x_n = (-1)^n$ so it follows that $b_1 b_2 \cdots b_n = 1$.

In the final step we will make a use of the well known AM-GM Inequality (Arithmetic Mean-Geometric Mean): $2 + b_i = 1 + 1 + b_1 \geq 3\sqrt[3]{1 \cdot 1 \cdot b_i} = 3\sqrt[3]{b_i}$. Hence

$p(2) = (2 + b_1)(2 + b_2) \cdots (2 + b_n) \geq 3\sqrt[3]{b_1} \cdot 3\sqrt[3]{b_2} \cdots 3\sqrt[3]{b_n} = 3^n \sqrt[3]{b_1 b_2 \cdots b_n} = 3^n.$

16. Putting $y = x - 1$ and $q(y) = p(y-1)$ yields $\big(p(x-2)\big)^2 = \big(p(y-1)\big)^2 = \big(q(y)\big)^2$ and $p(x^2 - 2x) = p(x^2 - 2x + 1 - 1) = p(y^2 - 1) = q(y^2)$. Hence the equality $p(x^2 - 2x) = \big(p(x-2)\big)^2$ transforms into $q(y^2) = \big(q(y)\big)^2$.

In order to find all polynomials $q(x)$ such that $q(y^2) = \big(q(y)\big)^2$ we can easily see that the polynomials $q(y) \equiv 0$ and $q(y) = y^n$ are possible solutions for all integers $n \geq 1$.

So let us assume that $q(y) = a_n y^n + a_{n-1} y^{n-1} + \cdots + a_1 y + a_0$ is a solution, where $a_n \neq 0$ and at least one of the coefficients $a_{n-1}, a_{n-1}, \ldots a_1, a_0$ is non-zero.

Let k be the largest integer such that $k < n$ and $a_k \neq 0$. Then $q(y^2) = a_n y^{2n} + a_k y^{2k} + \cdots + a_1 y^2 + a_0$, while $\big(q(y)\big)^2 = \big(a_n y^n + a_k y^k + \cdots + a_1 y + a_0\big)^2$.

If we now compare the coefficient at the y^{n+k} in both polynomials we get $0 = 2a_n a_k$, which contradicts the assumption. Thus $a_{n-1} = a_{n-1} = \cdots = a_1 = a_0 = 0$. Hence, $q(y) = a_n y^n$. From the condition $q(y^2) = \big(q(y)\big)^2$ we find that $a_n y^{2n} = a_n^2 y^{2n}$. Since this is valid for all y and $a_n \neq 0$, then $a_n = 1$. Thus only the polynomials $q(y) \equiv 0$ and $q(y) = y^n$ are possible solutions for all integers $n \geq 1$.

Finally $p(x) \equiv 0$ and $p(x) = q(x+1) = (x+1)^n$ are the only solutions to the original equation.

17. Consider the polynomial $p(x) = (x + a_1)(x + a_2) \cdots (x + a_n) - (x - b_1)(x - b_2) \cdots (x - b_n)$. It is clear that $\deg p(x) < n$.

For each $j = 1, 2, \ldots, n$, we have $p(b_j) = (b_j + a_1)(b_j + a_2) \cdots (b_j + a_n)$, i.e. the product of the numbers in the j-th column. According to the assumption, this product is the same, say c, for all columns.

Consider then the polynomial $q(x) = p(x) - c$. Since $\deg q(x) = \deg p(x) < n$ and, at the same time $q(x)$ has n distinct roots $b_1, b_2, \ldots, b_n$, then $q(x) \equiv 0$, i.e. $p(x) = c$ for all x.

Taking $x = -a_i$ (for $i = 1, 2, \ldots, n$) we get $c = p(-a_i) = -(-a_i - b_1)(-a_i - b_2) \cdots (-a_i - b_n) = (-1)^{n+1}(a_i + b_1)(a_i + b_2) \cdots (a_i + b_n)$, i.e. the product of the elements in the i-th row is $(-1)^{n+1}c$, independently of the row.

18. Consider the equation $x^3 - x - 33^{1992} = 0$. If it has a rational root β then it must be an integer. This is however impossible since $\beta^3 - \beta = \beta(\beta - 1)(\beta + 1)$ is then even, while 33^{1992} is odd.

Suppose it has a real root β (it is obvious that β is a rather large number, lat's say $\beta > 10$). Then $x^3 - x - 33^{1992} = (x - \beta)(x^2 + \gamma x + \delta) = x^3 + (\gamma - \beta)x^2 + (\delta - \beta\gamma)x - \beta\delta$. Identifying the coefficients on the left-hand side and the right-hand side of this equality gives $\gamma = \beta \;\; \delta = \beta^2 - 1$.

The discriminant of the equation $x^2+\gamma x+\delta = x^2+\beta x+(\beta^2-1) = 0$ is $\beta^2-4(\beta^2-1) = 4-3\beta^2$. Since $\beta > 10$ then the discriminant is negative. Hence the equation has no real roots.

Consequently the equation $x^3 - x - 33^{1992} = 0$ has only one real root. Since in the formulation of the problem it is said that both α and $p(\alpha)$ are real roots of this equation, then $\alpha = p(\alpha)$. Hence $p^n(\alpha) = \alpha$ for all integers $n \geq 1$ and the conclusion follows.

19. A quick glance at the given equality suggest that the polynomial $p_1(x) = x^2$ is one solution to the given functional equation. It is easy to verify that this is a case. Thus, let us introduce the polynomial $q(x) = p(x) - x^2$.

The substitution $p(x) = q(x)+x^2$ transforms given equation into $q(x) = \dfrac{q(x-1)+q(x+1)}{2}$, or $q(x) - q(x-1) = q(x+1) - q(x)$. If we then put $p_2(x) = q(x) - q(x-1)$ then we have $p_2(x) = p_2(x+1)$. Thus, for any fixed value x_0 we have $p_2(x_0) = p_2(x_0 + 1) = p_2(x_0 + 2) = p_2(x_0 + 3) = \ldots$.

Since $p_2(x)$ has infinitely many values in common with the constant polynomial $p_3(x) \equiv p_2(x_0)$, then $p_2(x)$ is constant, $p_2(x) \equiv b$ for some real number b. This means that $q(x) = q(x-1) + b$.

Since the last equation is satisfied by the linear function $p_4(x) = bx$ then we may consider instead a new polynomial $r(x) = q(x) - bx$.

The substitution $q(x) = r(x) + bx$ transforms the equation $q(x) = q(x-1) + b$ into $r(x) = r(x-1)$. As above in case of $p_2(x)$ we find that $r(x)$ is a constant polynomial, $r(x) \equiv c$ for some real number c.

Thus $q(x) = bx + c$ and finally, $p(x) = x^2 + bx + c$ for any choice of real constants b and c.

Now it only remains to verify that the polynomials $p(x) = x^2 + bx + c$ really satisfy the given equation.

20. Let us try the substitution $x = 2\cos t$, which transform the interval $[0, \pi]$ onto $[-2, 2]$. Using the formula $2\cos^2 t - 1 = \cos 2t$ we get

$p_1(x) = p_1(2\cos t) = 4\cos^2 t - 2 = 2\cos 2t$,
$p_2(x) = p_1\big(p_1(2\cos t)\big) = p_1(2\cos 2t) = 4\cos^2 2t - 2 = 2\cos 2^2 t$,
$p_3(x) = p_1\big(p_2(2\cos t)\big) = p_1(2\cos 2^2 t) = 4\cos^2 2^2 t - 2 = 2\cos 2^3 t$, and so on.

One can easily show by induction that $p_n(x) = 2\cos 2^n t$ for all positive integers n. The equation $p_n(x) = x$ may then be expressed as $2\cos 2^n t = 2\cos t$, i.e. $\cos 2^n t = \cos t$.

All solutions to the equation $\cos 2^n t = \cos t$ are $2^n t = \pm t + 2m\pi$, for $m \in \mathbb{Z}$, i.e. $t = \dfrac{2m\pi}{2^n - 1}$ and $t = \dfrac{2m\pi}{2^n + 1}$.

Since we only are interested in the solutions with $t \in [0, \pi]$, then, for $t = \dfrac{2m\pi}{2^n - 1}$ we must have $0 \le m \le 2^{n-1} - 1$, while for $t = \dfrac{2m\pi}{2^n + 1}$ we have $0 \le m \le 2^{n-1}$.

For $m_1, m_2 \neq 0$ the values $t = \dfrac{2m_1\pi}{2^n - 1}$ and $t = \dfrac{2m_2\pi}{2^n + 1}$ are distinct, for otherwise we would have $\dfrac{m_1}{2^n - 1} = \dfrac{m_2}{2^n + 1}$. This can be written as $\dfrac{m_1}{m_2} = \dfrac{2^n - 1}{2^n + 1}$. Adding 1 to both sides yields $\dfrac{m_1 + m_2}{m_2} = \dfrac{2^{n+1}}{2^n + 1}$, i.e. $(m_1 + m_2)(2^n + 1) = 2^{n+1} m_2$. Since 2^{n+1} divides the right-hand side and $2^n + 1$ and 2^{n+1} have no common factors then 2^{n+1} divides $m_1 + m_2$. This is however impossible because $m_1 + m_2 \le (2^{n-1} - 1) + 2^{n-1} = 2^n - 1 < 2^{n+1}$.

Thus we all together have 2^n distinct values of t corresponding to 2^n distinct values of $x = 2\cos t$.

21. Suppose, in the contrary, that $p(x) = q_1(x) \cdot q_2(x)$,where $\deg q_1(x), \deg q_2(x) > 0$ and both factors have integer coefficients. Having $p(0) = 3$, we may assume that $q_1(0) = \pm 1$, $q_2(0) = \mp 3$ and that $q_1(0) = -x^k + a_{k-1}x^{k-1} + \cdots + a_1 x \pm 1$.

Since $p(1)$, $p(-1) \neq 0$ then we must have $k > 1$ (for $k = 1$ one of the numbers $1, -1$ would be a zero of $q_1(x)$). So let $x_1, x_2, \ldots, x_k$ be the (complex) roots of $q_1(x)$).

Thus, $q_1(x) = (x - x_1)(x - x_2)\cdots(x - x_k)$ and, by the Viète's identities, $x_1 x_2 \cdots x_k = \pm 1$.

Note now that for each $i = 1, 2, \ldots, k$ we have $p(x_i) = x_i^n + 5x_i^{n-1} + 3$, i.e. $x_i^{n-1}(x_i + 5) = -3$. Multiplying all those expressions yields $\prod_{i=1}^{k} x_i^{n-1}(x_i + 5) = (-3)^k$. Using the equality $x_1 x_2 \cdots x_k = \pm 1$, the product above reduces to $(x_i + 5)(x_2 + 5)\cdots(x_k + 5) = (\pm 1)^{n-1}(-3)^k$ and we may write $\left|(x_i + 5)(x_2 + 5)\cdots(x_k + 5)\right| = 3^k$.

Let us find now the value of $\left|q_1(x)\right|$ for $x = -5$. We have $\left|q_1(-5)\right| = \left|(-5 - x_1)(-5 - x_2)\cdots(-5 - x_k)\right| = \left|(5 + x_1)(5 + x_2)\cdots(5 + x_k)\right| = 3^k \ge 9$.

However $q_1(-5)$ must be a divisor of $p(-5) = q_1(-5) \cdot q_2(-5)$, and since $p(-5) = 3$, we get a contradiction. Hence, $p(x)$ cannot be expressed as a product of two non-constant polynomials with integer coefficients.

22. Since the equation is reciprocal we may start by dividing the equation by x^2. This yields $x^2 + \dfrac{1}{x^2} + ax + a\dfrac{1}{x} + b = 0$. By taking now $y = x + \dfrac{1}{x}$, we will reduce the equation to ($\star$) $y^2 + ay + b - 2 = 0$.

In order to have at least one real root of the original equation we need that $y = x + \dfrac{1}{x}$, i.e. $x^2 - yx + 1 = 0$ has a real root x. For that we need a non-negative discriminant: $y^2 - 4 \ge 0$, i.e. $|y| \ge 2$.

The roots of the equation $(\star)$ are $y = \dfrac{-a \pm \sqrt{a^2-4b+8}}{2}$. We need at least one root y with $|y| \geq 2$. Hence $\dfrac{|a| + \sqrt{a^2-4b+8}}{2} \geq 2$.

At this point it would feel useful to square the inequality $\sqrt{a^2-4b+8} \geq 4-|a|$. To that end one would need an assumption that $4-|a| \geq 0$. This is however safe to assume that, because for $|a| = 4$, $a^2+b^2 \geq 16$. A closer look at the original equation shows that for $a = 1$ and $b = 0$ there are real roots, for example $x = -1$. Thus, the least possible value of a^2+b^2 is at most 1, and we man assume that $|a|$ and $|b|$ are at most 1.

Squaring the inequality $\sqrt{a^2-4b+8} \geq 4-|a|$yields $a^2-4b+8 \geq 16-8|a|+a^2$, i.e. $2|a|-b \geq 2$. Thus $-b \geq 2(1-|a|) \geq 0$.

Finally, $a^2+b^2 \geq a^2+4(1-|a|)^2 = 5a^2-8|a|+4 = 5(|a|-\frac{4}{5})^2+\frac{4}{5}$. The least possible value of a^2+b^2 is then $\frac{4}{5}$ and is achieved when $a = 0$ and $b = -\frac{2}{5}$ (follows from the equality $-b = 2(1-|a|) \geq 0$).

For these particular values of a and b the original equation takes form $x^4+\frac{4}{5}x^3-\frac{2}{5}x^2+\frac{4}{5}x+1 = (x+1)^2(x^2-\frac{6}{5}x+1) = 0$.

ADDITIONAL PROBLEMS

Here follow some more problems, this time without solutions offered. Instead, after the problems there are some hints and answers.

Problems.

23. Prove that $(x-1)^2$ is a divisor of $nx^{n+1}-(n+1)x^n+1$, for all $n \in \mathbb{N}$.

24. Let $p(x) = \sum_{k=0}^{n} a_k x^k$ and suppose that the line $y = a$ meets the graph of $p(x)$ in points $A_1, A_2, \ldots, A_n$. Suppose also that the line $y = a$ meets the graph of $p(x)$ in points $B_1, B_2, \ldots, B_n$. For $i = 1, 2, \ldots, n$ let the line A_iB_i form the angle α_i with the x-axis. Find $\cot\alpha_1 + \cot\alpha_2 + \cdots + \cot\alpha_n$.

25. Given $p(x) = \sum_{k=0}^{n} x^k$. Find the rest term $r(x)$ when dividning $p(x^n)$ by $p(x)$.

26. Prove that the polynomial $p(x) = x^{2n}-2x^{2n-1}+3x^{2n-2}-\cdots+(2n-1)x^2-2nx+(2n+1)$ has no real zeros.

27. Solve the following system of the equations: $\begin{cases} x+y+z & = & 2 \\ x^2+y^2+z^2 & = & 14 \\ x^3+y^3+z^3 & = & 20 \end{cases}$.

28. Is it possible that each of the polynomials $p_1(x) = ax^2 + bx + c$, $p_2(x) = cx^2 + ax + b$ and $p_3(x) = bx^2 + cx + a$ has two real zeros?

29. (Polish-Austrian, 1988) Let $p(x) \in \mathbb{Z}$ be a polynomial such that $q(x) = p(x) + 12$ has at least six distinct integer zeros. Show that $p(x)$ has no integer zeros.

30. Let $p(x) \in \mathbb{Z}[x]$ and suppose that $p(23) \cdot p(32)$ is an odd number. Show that $p(x)$ has no integer zeros.

31. (Bulgaria, 1971) Let a be an integer not divisible by 5. Show that $p(x) = x^5 - x + a$ is irreducible in $\mathbb{Z}[x]$ (i.e. cannot be written as a product of two non-constant polynomials in $\mathbb{Z}[x]$).

32. (India, 1989) Prove that if the polynomial $p(x) = x^6 + ax^3 + bx^2 + cx + d$ has all zeros real then $a = b = c = d = 0$.

33. (India, 1989) Let $p(x) = x^4 + 26x^3 + 52x^2 + 78x + 1989$. Prove that $p(x)$ can not be written as a product of two polynomials, $p(x) = q_1(x)q_2(x)$, both with integer coefficients and degree less than 4.

34. (Austria, 1990) Find all values of the rational number a for which the zeros of the polynomial $p(x) = ax^2 + (a+1)x + a - 1$ are integers.

35. (Russia, 1993) Suppose that $p(x) \in \mathbb{Z}[x]$ satisfies $p(1993) \cdot p(1994) = 1995$. Does $p(x)$ have integer zeros?

36. (Hungary, 1987) Let $p(x) = ax^3 + bx^2 + cx + d \in \mathbb{Q}[x]$ be such that one of its zeros, x_1, is the product of the two others. Show that:

(1) If $a = 1$ and $b \neq 1$ then x_1 is rational.

(2) If $b = 0$ then x_1 is rational.

37. (Netherlands, 1990) The polynomial $p(x) = ax^4 + bx^3 + cx^2 + dx$ satisfies the following conditions:

(1) $a, b, c, d > 0$, (2) $p(x)$ is an integer for $x = \pm 1, \pm 2$, (3) $p(1) = 1$ and $p(5) = 70$.

Find a, b, c, d and show that $p(x)$ is an integer for all $x \in \mathbb{Z}$.

38. (Poland, 1972) Let $p(x) \in \mathbb{Z}[x]$ and suppose that, for some integer k, $p(k)$, $p(k+1)$ and $p(k+2)$ are divisible by 3. Prove that $p(x)$ is divisible by 3 for every integer x.

39. (Putnam, 1968) Find all monic polynomials $p(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$ with all coefficients equal -1 or 1 and having all zeros real.

40. (Belarus, 1994) Determine all pairs of monic polynomials $\big(p(x), q(x)\big)$ such that $p\big(q(x)\big) = x^{1994}$.

41. (Romania, 1962) Prove that for each real number α and each positive integer n the polynomial $p(x) = (\sin\alpha)x^n - (\sin n\alpha)x + \sin(n-1)\alpha$ is divisible by $q(x) = x^2 - (2\cos\alpha)x + 1$.

42. (Hong Kong, 1994) Let $p(x) \in \mathbb{Z}[x]$ be a monic polynomial of degree 1991. Prove that $q(x) = \big(p(x)\big)^2 - 9$ has at most 1995 distinct integer zreros.

43. (Canada, 1974) Suppose that the coefficients of the polynomial $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in \mathbb{R}[x]$ satisfy $0 \leq a_1 \leq a_0$ for $i = 1, 2, \ldots, n$. Prove that if $\big(p(x)\big)^2 = b_{2n}x^{2n} + b_{2n-1}x^{2n-1} + \cdots + b_{n+1}x^{n+1} + \cdots + b_1x + b_0$, then $b_{n+1} \leq \frac{1}{2}\big(p(1)\big)^2$.

44. (Sovjet, 1990) Let $p(x) = ax^2 + bx + c \in \mathbb{R}[x]$ with $a, b, c > 0$ and $a + b + c = 1$. Prove that if $x_1x_2\cdots x_n = 1$ for n positive reals $x_1, x_2, \ldots, x_n$ then $p(x_1)p(x_2)\cdots p(x_n) \geq 1$.

Hints and answers.

23. Hint: Multiple roots.

24. Hint: Use the first of Viète's identities.

Answer: 0.

25. Hint: Start by dividing $p(x)$ by $(x-1)$. Note also that $x^n - 1 = (x-1)p(x)$.

Answer: $r(x) \equiv n$.

26. Hint: Show that no zeros can be negative. Then consider $p(x) + xp(x)$.

27. Hint: Consider a third degree polynomial with the roots being the solutions of the system. Compare with the solution of Problem 4.

Answer: Six solutions being all permutations of the numbers $1, -2, 3$.

28. Hint: Use the discriminats: the discriminat for $p(x) = \alpha x^2 + \beta x + \gamma$ is the number $\beta^2 - 4\alpha\gamma$.

Answer: No.

29. Hint: Since $q(x) = (x-\alpha_1)\cdots(x-\alpha_6)r(x)$, where all α_k are distinct integers and $r(x) \in \mathbb{Z}[x]$, consider $q(x_0)$ for the zero x_0 of $p(x)$.

30. Hint: If a is an integer then $p(x) = (x-a)q(x)$ where $q(x) \in \mathbb{Z}[x]$. What can be said about the parity of $p(23) \cdot p(32)$?

31. Hint: One has to consider two cases: $p(x) = (x-b)q(x)$ and $p(x) = (x^2 - bx - c)q(x)$. In both cases the conclusion will be that 5 divides a.

32. Hint: What can be said about the sum of the squares of the zeros of $p(x)$?

33. Hint: Consider two cases: (1) factors of degree 1 and 3, and (2) factors of degree 2 and 2. Investigate then divisibility of coefficients by 13.

34. Hint: Viète's identities.

Answer: $a = 0,\ 1$ and $-\dfrac{1}{7}$.

35. Hint: Assume that k is a integer zero of $p(x)$ and write then $p(x) = (x - k)q(x)$, where $q(x)$ has then integer coefficients.

Answer: No!

36. Hint: Viète's identities.

Answer: (1) $x_1 = \dfrac{c-d}{1-b}$, (2) $x_1 = \dfrac{c-d}{a}$.

37. Hint: Divisibility.

38. Hint: use the fact that $p(k_1) - p(k_2)$ is divisible by $k_1 - k_2$ for any two distinct integers k_1, k_2.

39. Hint: Use the Viète's identities and count the sum of the squares of the zeros of $p(x)$. With a help of AM-GM Inequality (applied to this sum) show that $n \leq 3$. Then find the polynomials.

Answer: Six polynomials: $(x^2 - 1)(x - 1)$, $(x^2 - 1)(x + 1)$, $x^2 - x - 1$, $x^2 + x - 1$, $x - 1$ and $x + 1$.

40. Hint: Note that $\deg p\big(q(x)\big) = \deg p(x) \cdot \deg q(x)$ and that $1994 = 1 \cdot 1994 = 2 \cdot 997$ are the only factorizations of 1994. Hence there are four cases to be considered.

Answer: The following pairs are the solutions to the problem: (1) $p(x) = x + a$, $q(x) = x^{1994} - a$, (2) $p(x) = (x - a)^{1994}$, $q(x) = x + a$, (3) $p(x) = (x + a)^2$, $q(x) = x^{997} - a$ and (4) $p(x) = (x - a)^{997}$, $q(x) = x^2 + a$, where in each case a is an arbitrary constant.

41. Hint: Use the induction and some trigonometrical identities. Alternatively use complex numbers: by taking $z = \cos\alpha + i\sin\alpha$, show that $q(x) = (x - z)(x - \overline{z})$. Use then de Moivre's Theorem staying that $(\cos\alpha + i\sin\alpha)^n = \cos n\alpha + i\sin n\alpha$, for each $n \in \mathbb{Z}$.

42. Hint: Let $a_1, a_2, \ldots, a_n$ be all distinct, integer roots of the equation $p(x) = 3$. The we have $p(x) = (x - a_1)(x - a_2)\cdots(x - a_m)p_1(x) + 3$ for some (monic) polynomial $p_1(x)$. Let b be an integer roots of $p(x) = -3$. Consider $p(b)$ and find out that $m \leq 4$.

43. Hint: $b_{n+1} = a_1a_n + a_2a_{n-1} + \cdots + a_na_1$ and $\big(p(1)\big)^2 = \big(a_1 + a_2 + \cdots + a_n\big)$.

44. Hint: Start by showing that $p(x) \cdot p(y) \geq \big(p(\sqrt{xy})\big)^2$. Later you may have a use of the fact that

$p(1) = 1.$