THE UNDENIABLE CHARM OF INEQUALITIES

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A Short Guided Tour through the Jungle of Algebraic Inequalities

INTRODUCTION

What follows is a list of eleven most useful algebraic inequalities with a large collection of examples and related problems. The inequalities are presented in a simple form: they may all be strengthened and generalized. The choice of formulation is made in accordance with usefulness for solving problems in mathematical competitions. Moreover, the inequalities presented here are in no way "independent". For example, Hölder's inequality is a consequence of Jensen's; both Cauchy-Schwarz and Chebyshev's inequalities may be derived from the Rearrangement inequality, and so on. Nevertheless, because of their applicability all these inequalities deserve to be stated separately.

The inequalities presented are the following

1. AM-GM-HM inequality
2. Chebyshev's inequality
3. Rearrangement inequality
4. Cauchy-Schwarz inequality

5. Hölder's inequality
6. Minkowski's inequality
7. Jensen's inequality
8. Power Mean inequality
9. Schur's inequality
10. Maclaurin's inequality
11. Muirhead's inequality

INEQUALITIES

1. AM-GM-HM inequality

Let $a_1, a_2, ..., a_n$ be positive real numbers. Then

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{n}{1/a_1 + 1/a_2 + \cdots + 1/a_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$. □

This probably is the best-known inequality and is very useful in various situations. At the same time it is only a special case of several other inequalities listed below.

Example 1. (UK, 2000) Given that x, y, z are positive real numbers satisfying $xyz = 32$, find the minimum value of $x^2 + 4xy + 4y^2 + 2z^2$.

Solution. Applying the AM-GM inequality twice, we find that $x^2 + 4xy + 4y^2 + 2z^2 = (x^2 + 4y^2) + 4xy + 2z^2 \geq 2\sqrt{x^2 \cdot 4y^2} + 4xy + 2z^2 = 4xy + 4xy + 2z^2 \geq 3\sqrt[3]{32x^2y^2z^2} = 3\sqrt[3]{32^3} = 96$.
Equality holds when $x^2 = 4y^2$ and $4xy = 2z^2$, it is, when $x = z = 4$ and $y = 2$. □

Example 2. (IMO, 1964) Let a, b, c be sides of a triangle. Prove that $a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c) \leq 3abc$.

Solution. Let $x = a + b - c$, $y = b + c - a$, $z = c + a - b$. Then $x, y, z > 0$ and the inequality wa have to prove becomes $\left(\frac{z + x}{2}\right)^2 y + \left(\frac{x + y}{2}\right)^2 z + \left(\frac{y + z}{2}\right)^2 x \leq \frac{3}{8}(z + x)(x + y)(y + z)$.

This inequality reduces to $x^2y + xy^2 + x^2z + xz^2 + y^2z + yz^2 \geq 6xyz$, which is true because $x^2y + xy^2 + x^2z + xz^2 + y^2z + yz^2 \geq 6\sqrt[6]{x^6y^6z^6} = 6xyz$ by the AM-GM inequality. The equality occurs if and only if $x = y = z$, i.e. if and only if the triangle is equilateral. □

Example 3. Given that the equation $x^4 + px^3 + qx^2 + rx + s = 0$ has four real positive roots, prove that

$$(i) \qquad pr - 16s \geq 0,$$

$$(ii) \qquad q^2 - 36s \geq 0.$$

Solution. Suppose x_1, x_2, x_3 and x_4 are the roots of the equation. Then, we have $x_1+x_2+x_3+x_4 = -p$, $x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4 = q$, $x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4 = -r$ and $x_1x_2x_3x_4 = s$.

Using the AM-HM inequality we get $pr = \Big(\sum_{i=1}^{4} x_i\Big)\Big(\sum_{i=1}^{4} \frac{1}{x_i}\Big)x_1x_2x_3x_4 \geq 16s$. From the AM-GM inequality we have $q = x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4 \geq 6(x_1^3x_2^3x_3^3x_4^3)^{1/6} = 6s^{1/2}$. □

Example 4. (IMO, 1999) Let $n \geq 2$ be a fixed integer. Find the smallest constant C such that for all non-negative reals $x_1, ..., x_n$:

$$\sum_{1\leq i<j\leq n} x_ix_j(x_i^2+x_j^2) \leq C\Big(\sum_{i=1}^{n} x_i\Big)^4.$$

Determine when equality occurs.

Solution. We give here a surprise-solution supplied by one of the Chinese contestants *after* the competition. It requires only one, although tricky, use of the AM-GM inequality.

$$\Big(\sum_{i=1}^{n} x_i\Big)^4 = \Big(\sum_{i=1}^{n} x_i^2 + 2\sum_{1\leq i<j\leq n} x_ix_j\Big)^2 \geq 4\Big(\sum_{k=1}^{n} x_k^2\Big)\Big(2\sum_{1\leq i<j\leq n} x_ix_j\Big) =$$

$$8\Big(\sum_{1\leq i<j\leq n} x_ix_j \sum_{k=1}^{n} x_k^2\Big) \geq 8\sum_{1\leq i<j\leq n} x_ix_j(x_i^2+x_j^2).$$

The second inequality is an equality if and only if $n-2$ of the x_i's are zeros. Let us therefore then assume that $x_3 = x_4 = \cdots = x_n = 0$. Then, for the first inequality to become an equality one requires that $(x_1^2+x_2^2+2x_1x_2)^2 = 8(x_1^2+x_2^2)x_1x_2$, which reduces to $(x_1-x_2)^4 = 0$. Hence $x_1 = x_2$. The answer is $C = \frac{1}{8}$ with equality if and only if two of the x_i's are equal and the rest are zeros. □

2. Chebyshev's inequality

Let $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ be two sequences of real numbers, at least one of which consists entirely of positive numbers. Assume that $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$. Then

$$\frac{1}{n}(a_1+a_2+\cdots+a_n) \cdot \frac{1}{n}(b_1+b_2+\cdots+b_n) \leq \frac{1}{n}(a_1b_1+a_2b_2+\cdots+a_nb_n).$$

If we instead assume that $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$, then the inequality is reversed. Equality occurs if and only if at least one of the sequences is constant. □

Example 5. Let $a_1, a_2, ..., a_n$ be a sequence of positive real numbers with the arithmetical mean A. Prove that $\frac{1}{n}\sum_{k=1}^{n}\Big(a_k + \frac{1}{a_k}\Big)^2 \geq \Big(A + \frac{1}{A}\Big)^2$.

Solution. By squaring we get the equivalent inequality $\frac{1}{n}\sum_{k=1}^{n} a_k^2 + \frac{1}{n}\sum_{k=1}^{n}\Big(\frac{1}{a_k^2}\Big) \geq A^2 + \Big(\frac{1}{A}\Big)^2$. We will show it in two steps.

We can, without loss of generality, assume that the sequence $a_1, a_2, ..., a_n$ is increasing. Then the sequence $1/a_1, 1/a_2, ..., 1/a_n$ is decreasing and, by using Chebyshev's inequality twice, we get
$\frac{1}{n}\sum_{k=1}^{n}\Big(\frac{1}{a_k^2}\Big)A^2 =$
$\frac{1}{n}\sum_{k=1}^{n}\Big(\frac{1}{a_k^2}\Big)\Big(\frac{1}{n}\sum_{k=1}^{n} a_k\Big)\Big(\frac{1}{n}\sum_{k=1}^{n} a_k\Big) \geq \Big(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{a_k}\Big)\Big(\frac{1}{n}\sum_{k=1}^{n} a_k\Big) \geq \frac{1}{n}\sum_{k=1}^{n} 1 = 1.$
From Chebyshev's inequality we get also $\frac{1}{n}\sum_{k=1}^{n} a_k^2 \geq \Big(\frac{1}{n}\sum_{k=1}^{n} a_k\Big)\Big(\frac{1}{n}\sum_{k=1}^{n} a_k\Big)$
$= A^2.$
These two results together give the inequality in question. □

Example 6. Let $b_1, b_2, ..., b_n$ be positive real numbers. Show that

$$\Big(\sum_{i=1}^{n}\frac{1}{b_i}\Big)^2 \sum_{i=1}^{n} {b_i}^2 \geq n^3.$$

Solution. We may write the desired inequality as $\Big[(\sum_{i=1}^{n}\frac{1}{b_i})/n\Big]\cdot\Big[(\sum_{i=1}^{n}\frac{1}{b_i})/n\Big]\cdot\Big[(\sum_{i=1}^{n}{b_i}^2)/n\Big] \geq 1$. Because of the symmetry we may assume that the sequence $b_1, b_2, ..., b_n$ is increasing and then we notice that the sequence $\frac{1}{b_1}, \frac{1}{b_2}, ..., \frac{1}{b_n}$ is decreasing. Using Chebyshev's inequality twice, first with the second and the third factor, we get $\Big[(\sum_{i=1}^{n}\frac{1}{b_i})/n\Big]\cdot\Big[(\sum_{i=1}^{n}\frac{1}{b_i})/n\Big]\cdot\Big[(\sum_{i=1}^{n}{b_i}^2)/n\Big] \geq$ $\Big[(\sum_{i=1}^{n}\frac{1}{b_i})/n\Big]\cdot\Big[(\sum_{i=1}^{n} b_i)/n\Big] \geq (\sum_{i=1}^{n} 1)/n = 1.$ □

Example 7. (India, 1995) Let n be an integer greater than 1 and let $a_1, a_2, ..., a_n$ be positive real numbers with the sum 1. Show that

$$\frac{a_1}{\sqrt{1-a_1}} + \frac{a_2}{\sqrt{1-a_2}} + \cdots + \frac{a_n}{\sqrt{1-a_n}} \geq \sqrt{\frac{n}{n-1}}.$$

Solution. We may assume that the sequence $a_1, a_2, ..., a_n$ is increasing and then the sequence $\frac{1}{\sqrt{1-a_1}}, \frac{1}{\sqrt{1-a_2}}, ..., \frac{1}{\sqrt{1-a_n}}$ is increasing as well. Chebyshev's inequality now implies that $\frac{a_1}{\sqrt{1-a_1}} +$

$\frac{a_2}{\sqrt{1-a_2}} + \cdots + \frac{a_n}{\sqrt{1-a_n}} \geq \frac{1}{n}(a_1 + a_2 + ... + a_n)(\frac{1}{\sqrt{1-a_1}} + \frac{1}{\sqrt{1-a_2}} + \cdots + \frac{1}{\sqrt{1-a_n}}) = \frac{1}{n}(\frac{1}{\sqrt{1-a_1}} + \frac{1}{\sqrt{1-a_2}} + \cdots + \frac{1}{\sqrt{1-a_n}})$.

Now we can use the result from the previous example. Letting $b_i = \frac{1}{\sqrt{1-a_i}}$, we get $\Big(\frac{1}{n}\sum_{i=1}^{n} b_i\Big)^2 \geq n\Big(\sum_{i=1}^{n} {b_i}^2\Big)^{-1} = n\Big(\sum_{i=1}^{n}(1-a_i)\Big)^{-1} = \frac{n}{n-1}$.

Hence $\frac{1}{n}(\frac{1}{\sqrt{1-a_1}} + \frac{1}{\sqrt{1-a_2}} + \cdots + \frac{1}{\sqrt{1-a_n}}) \geq \sqrt{\frac{n}{n-1}}$.

The equality holds if and only if $a_1 = a_2 = \cdots = a_n = \frac{1}{n}$. □

3. Rearrangement inequality

Let $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ be real numbers. For any permutation $(a_1^{'}, a_2^{'}, ..., a_n^{'})$ of $(a_1, a_2, ..., a_n)$, we have

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \geq a_1^{'}b_1 + a_2^{'}b_2 + \cdots + a_n^{'}b_n \geq a_nb_1 + a_{n-1}b_2 + \cdots + a_ab_n,$$

with equality if and only if the sequence $(a_1^{'}, a_2^{'}, ..., a_n^{'})$ is equal to $(a_1, a_2, ..., a_n)$ or $(a_n, a_{n-1}, ..., a_1)$ respectively. □

With the notation $\begin{bmatrix} a_1 & a_2 & ... & a_n \\ b_1 & b_2 & ... & b_n \end{bmatrix} = a_1b_1 + a_2b_2 + \cdots a_nb_n$ the statement of the inequality above may be written as

$$\begin{bmatrix} a_1 & a_2 & ... & a_n \\ b_1 & b_2 & ... & b_n \end{bmatrix} \geq \begin{bmatrix} a_1^{'} & a_2^{'} & ... & a_n^{'} \\ b_1 & b_2 & ... & b_n \end{bmatrix} \geq \begin{bmatrix} a_n & a_{n-1} & ... & a_1 \\ b_1 & b_2 & ... & b_n \end{bmatrix}$$

This innocent-looking and easy to prove inequality is in fact a very powerful instrument and several other inequalities here can easily be derived from it. The Rearrangemant inequality is one of my two favourites. The second one is the Muirhead's inequality.

Example 8. (IMO 1975) Let x_i, y_i, $1 \leq i \leq n$, be real numbers such that $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$. Let $z_1, z_2, ..., z_n$ be any permutation of $y_1, y_2, ..., y_n$. Show that $\sum_{i=1}^{n}(x_i - y_i)^2 \leq \sum_{i=1}^{n}(x_i - z_i)^2$.

Solution. After squaring and cancelling equal terms, the inequality becomes $\sum_{i=1}^{n} x_iy_i \geq \sum_{i=1}^{n} x_iz_i$ which basically is the Rearrangement inequality. □

Example 9. For all positive numbers a, b and c, prove the inequality

$$\frac{a^3b}{c}+\frac{a^3c}{b}+\frac{b^3a}{c}+\frac{b^3c}{a}+\frac{c^3a}{b}+\frac{c^3b}{a}\geq 6abc$$

Solution. Because of the symmetry we may assume that $a \geq b \geq c$. Then the Rearrangement inequality implies that $\frac{a^3b}{c}+\frac{b^3c}{a}+\frac{c^3a}{b}=\begin{bmatrix} a^2b^2 & b^2c^2 & c^2a^2 \\ \frac{a}{bc} & \frac{b}{ca} & \frac{c}{ab} \end{bmatrix} \geq \begin{bmatrix} a^2b^2 & b^2c^2 & c^2a^2 \\ \frac{c}{ab} & \frac{a}{bc} & \frac{b}{ca} \end{bmatrix} = 3abc$ and, similarly, $\frac{a^3c}{b}+\frac{b^3a}{c}+\frac{c^3b}{a}=\begin{bmatrix} a^2b^2 & b^2c^2 & c^2a^2 \\ \frac{b}{ca} & \frac{c}{ab} & \frac{a}{bc} \end{bmatrix} \geq \begin{bmatrix} a^2b^2 & b^2c^2 & c^2a^2 \\ \frac{c}{ab} & \frac{a}{bc} & \frac{b}{ca} \end{bmatrix} = 3abc$. Adding these two inequalities together yields the desired result. □

Example 10. For all positive numbers a, b and c prove the inequality

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\geq\frac{a+b+c}{2}.$$

Solution. Because of the symmetry we may assume $a \geq b \geq c$. Then $\begin{bmatrix} a^2 & b^2 & c^2 \\ \frac{1}{b+c} & \frac{1}{c+a} & \frac{1}{a+b} \end{bmatrix} \geq \begin{bmatrix} a^2 & b^2 & c^2 \\ \frac{1}{c+a} & \frac{1}{a+b} & \frac{1}{b+c} \end{bmatrix}$ and $\begin{bmatrix} a^2 & b^2 & c^2 \\ \frac{1}{b+c} & \frac{1}{c+a} & \frac{1}{a+b} \end{bmatrix} \geq \begin{bmatrix} a^2 & b^2 & c^2 \\ \frac{1}{a+b} & \frac{1}{b+c} & \frac{1}{c+a} \end{bmatrix}$.

After adding those two inequalities and using the easy to prove inequality $\frac{x^2+y^2}{x+y}\geq\frac{x+y}{2}$, for positive x, y, the result follows. □

Example 11. (IMO, 1983) Let a, b, c be the sides of a triangle. Prove that

$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a)\geq 0.$$

Solution. We may assume $a \geq \max\{b, c\}$. If $a \geq b \geq c$, we first prove that $a(b+c-a) \leq b(c+a-b) \leq c(a+b-c)$. Note that $b(c+a-b)-a(b+c-a)=(a-b)(a+b-c)\geq 0$. The second inequality reduces to $(b-c)(b+c-a)\geq 0$ and is obvious by the triangle inequality.

Dividing the desired inequality by abc we have to show that $\frac{1}{c}a(a-b)+\frac{1}{a}b(b-c)+\frac{1}{b}c(c-a)\geq 0$, or, after subtracting $a+b+c$, that $\frac{1}{c}a(-c+a-b)+\frac{1}{a}b(-a+b-c)+\frac{1}{b}c(-b+c-a)\geq -(a+b+c)$. The desired inequality can be written as $\frac{1}{c}a(c-a+b)+\frac{1}{a}b(a-b+c)+\frac{1}{b}c(b-c+a)\leq a+b+c$.

By the Rearrangement inequality $\frac{1}{c}a(c-a+b)+\frac{1}{a}b(a-b+c)+\frac{1}{b}c(b-c+a)=$

$\begin{bmatrix} a(c-a+b) & b(a-b+c) & c(b-c+a) \\ \frac{1}{c} & \frac{1}{a} & \frac{1}{b} \end{bmatrix} \leq$

$$\left[\begin{array}{ccc} a(c-a+b) & b(a-b+c) & c(b-c+a) \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \end{array}\right] = a+b+c.$$

If $a \geq c \geq b$ the proof is similar. □

4. Cauchy-Schwarz inequality

For any real numbers $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$,

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \leq (a_1^2 + a_2^2 + \cdots + a_n^2) \cdot (b_1^2 + b_2^2 + \cdots + b_n^2),$$

with equality if and only if there are two real numbers α and β, not both equal to 0, such that $\alpha a_k = \beta b_k$ for all $k = 1, 2, ..., n$. □

Example 12. (Iran, 1998) Let $x, y, z > 1$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$. Prove that

$$\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

Solution. Observe that by hypothesis $\frac{x-1}{x} + \frac{y-1}{y} + \frac{z-1}{z} = 1$. Then, by using the Cauchy-Schwarz inequality, we get

$x+y+z = \left(\sqrt{x}^2 + \sqrt{y}^2 + \sqrt{z}^2\right)\left(\sqrt{\frac{x-1}{x}}^2 + \sqrt{\frac{y-1}{y}}^2 + \sqrt{\frac{z-1}{z}}^2\right) \geq \left(\sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}\right)^2$, which gives the desired inequality.

Example 13. (Romania, 1999) Let $n \geq 2$ be a positive integer and let $x_1, x_2, ..., x_n, y_1, y_2, ..., y_n$ be positive real numbers such that $x_1 + x_2 + \cdots + x_n \geq x_1y_1 + x_2y_2 + \cdots x_ny_n$. Prove that $x_1 + x_2 + \cdots + x_n \leq \frac{x_1}{y_1} + \frac{x_2}{y_2} + \cdots \frac{x_n}{y_n}$.

Solution. Applying the Cauchy-Schwarz inequality and then the given inequality, we have $\left(\sum_{i=1}^{n} x_i\right)^2 \leq \sum_{i=1}^{n} x_iy_i \cdot \sum_{i=1}^{n} \frac{x_i}{y_i} \leq \sum_{i=1}^{n} x_i \cdot \sum_{i=1}^{n} \frac{x_i}{y_i}$. Dividing both sides by $\sum_{i=1}^{n} x_i$ yields the desired inequality. □

Example 14. (USSR, 1986) Let $a_1, a_2, ..., a_n$ be positive real numbers. Prove that

$$\frac{1}{a_1} + \frac{2}{a_1+a_2} + \frac{3}{a_1+a_2+a_3} + \cdots + \frac{n}{a_1+a_2+\cdots+a_n} < 2\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

Solution. Let $S_k = \sum_{i=1}^{k} a_i$ and $A_k = \sum_{i=1}^{k} \frac{i^2}{a_i}$, for $k = 1, 2, ..., n$. Using the Cauchy-Schwarz inequality we now get $\left(\frac{k(k+1)}{2}\right)^2 = \left(\sum_{i=1}^{k} \frac{i}{\sqrt{a_i}} \cdot \sqrt{a_i}\right)^2 \leq \sum_{i=1}^{k} \frac{i^2}{a_i} \cdot \sum_{i=1}^{k} a_i = A_k S_k.$

Hence, $\frac{k}{S_k} \leq \frac{4kA_k}{k^2(k+1)^2} < \frac{(4k+2)A_k}{k^2(k+1)^2} = \left(\frac{2}{k^2} - \frac{2}{(k+1)^2}\right) A_k$. By adding the left-hand sides for all k, we receive
$\sum_{k=1}^{n} \frac{k}{S_k} \leq \sum_{k=1}^{n} \frac{2}{k^2} A_k - \sum_{k=2}^{n+1} \frac{2}{k^2} A_{k-1} = \frac{2}{a_1} + \sum_{k=2}^{n} \frac{2}{k^2}(A_k - A_{k-1}) - \frac{2}{(n+1)^2} A_n = 2\sum_{k=1}^{n} \frac{1}{a_k} - \frac{2}{(n+1)^2} A_n < 2\sum_{k=1}^{n} \frac{1}{a_k}.$ □

5. Hölder's inequality

Let p and q be two positive real numbers whose sum is 1 and let $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ be two sequences of positive real numbers. Then

$$\sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} a_k^{1/p}\right)^p \cdot \left(\sum_{k=1}^{n} b_k^{1/q}\right)^q,$$

with the equality if and only if there are two real numbers α and β, not both equal to 0, such that $\alpha a_k^{1/p} = \beta b_k^{1/q}$ for all $k = 1, 2, ..., n$.

This inequality can easily be generalized to more than two sequences:
Suppose $\{a_{i1}, a_{i2}, ..., a_{in}\}$, $i = 1, 2, ..., k$, are k sequences of positive real numbers and $p_1, p_2, ..., p_k$ are positive real numbers whose sum is 1. Then,

$$\sum_{j=1}^{n} a_{1j} a_{2j} ... a_{kj} \leq \left(\sum_{j=1}^{n} a_{1j}^{1/p_1}\right)^{p_1} \cdot \left(\sum_{j=1}^{n} a_{2j}^{1/p_2}\right)^{p_2} \cdot ... \cdot \left(\sum_{j=1}^{n} a_{kj}^{1/p_k}\right)^{p_k}.$$

□

Example 15. (Belarus, 2000) For all positive real numbers a, b, c, x, y, z prove that

$$\frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z} \geq \frac{(a+b+c)^3}{3(x+y+z)}.$$

Solution. We use the Hölder inequality generalized to three sequences: $p_1p_2p_3 + q_1q_2q_3 + r_1r_2r_3 \leq \prod_{i=1}^{3}(p_i^3 + q_i^3 + r_i^3)^{1/3}$, for all positive real numbers p_i, q_i, r_i.

Hence, $\left(\frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z}\right)^{1/3} (1+1+1)^{1/3} (x+y+z)^{1/3} \geq a+b+c$. Cubing both sides and then dividing by $3(x+y+z)$ give the desired result. □

Example 16. Let a, b, c, d be positive real numbers. Prove that

$$a^6b^3 + b^6c^3 + c^6d^3 + d^6a^3 \geq a^2b^5c^2 + b^2c^5d^2 + c^2d^5a^2 + d^2a^5b^2.$$

Solution. Let $x = (a^2b)^3$, $y = (b^2c)^3$, $z = (c^2d)^3$ and $t = (d^2a)^3$. Then $a^6b^3 + b^6c^3 + c^6d^3 + d^6a^3 = \left((x + y + z + t)^{1/3}\right)^3 = (x + y + z + t)^{1/3}(y + z + t + x)^{1/3}(y + z + t + x)^{1/3} \geq \left(xy^2\right)^{1/3} + \left(yz^2\right)^{1/3} + \left(zt^2\right)^{1/3} + \left(tx^2\right)^{1/3} = a^2b^5c^2 + b^2c^5d^2 + c^2d^5a^2 + d^2a^5b^2$. □

6. Minkowski's inequality

Given a real number $r \geq 1$ and two sequences of positive real numbers, $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$, we have

$$\left(\sum_{i=1}^{n}(a_i + b_i)^r\right)^{1/r} \leq \left(\sum_{i=1}^{n} a_i^r\right)^{1/r} + \left(\sum_{i=1}^{n} b_i^r\right)^{1/r},$$

with equality if and only if there are two real numbers α and β, not both equal 0, such that $\alpha a_k = \beta b_k$ for all $k = 1, 2, ..., n$.

For $0 < r < 1$ the inequality is reversed. □

It is again obvious that this inequality may be generalized to more than two sequences of positive real numbers.

Example 17. For all nonnegative real numbers x, y, z prove that

$$\sqrt{x + y} + \sqrt{y + z} + \sqrt{z + x} \geq \sqrt{2}(\sqrt{x} + \sqrt{y} + \sqrt{z}).$$

Solution. By taking $r = 2$ and letting $a_1 = \sqrt{x}$, $a_2 = \sqrt{y}$, $a_3 = \sqrt{z}$ we may use Minkowski's inequality in an obvious way: $\left(a_1^2 + a_2^2\right)^{1/2} + \left(a_2^2 + a_3^2\right)^{1/2} + \left(a_3^2 + a_1^2\right)^{1/2} \geq \left(\left(a_1 + a_2 + a_3\right)^2 + \left(a_2 + a_3 + a_1\right)^2\right)^{1/2} = \sqrt{2}(a_1 + a_2 + a_3)$. □

Example 18. Let $a_1, a_2, ..., a_n$ be a sequence of positive real numbers with the product $a_1a_2 \cdots a_n = 1$. Suppose that $(b_1, b_2, ..., b_n)$, $(c_1, c_2, ..., c_n)$ and $(d_1, d_2, ..., d_n)$ are three permutations of the sequence $(a_1, a_2, ..., a_n)$. Prove that

$$\sum_{k=1}^{n} \sqrt{a_k + b_k + c_k + d_k} \geq 2n.$$

Solution. By Minkowski's inequality $\left(\sum_{k=1}^{n} \sqrt{a_k + b_k + c_k + d_k}\right)^2 \geq \left(\sum_{k=1}^{n} \sqrt{a_k}\right)^2 + \left(\sum_{k=1}^{n} \sqrt{b_k}\right)^2 +$

$\Big(\sum_{k=1}^{n}\sqrt{c_k}\Big)^2+\Big(\sum_{k=1}^{n}\sqrt{d_k}\Big)^2=4\Big(\sum_{k=1}^{n}\sqrt{a_k}\Big)^2.$

By the AM-GM inequality $\sum_{k=1}^{n}\sqrt{a_k}\ge n\sqrt[n]{\sqrt{a_1}\sqrt{a_2}\cdots\sqrt{a_n}}=$ $n\sqrt[2n]{a_1a_2\cdots a_n}=n$. Those two inequalities together imply $\sum_{k=1}^{n}\sqrt{a_k+b_k+c_k+d_k}\ge 2n$. □

7. Jensen's inequality

Let $f(x)$ be a strictly convex function on an interval I and let $\alpha_1,\alpha_2,...,\alpha_n$ be nonnegative numbers such that $\sum_{k=1}^{n}\alpha_k=1$. Then, for all $x_1,x_2,...,x_n\in I$,

$$f(\alpha_1x_1+\alpha_2x_2+\cdots+\alpha_nx_n)\le\alpha_1f(x_1)+\alpha_2f(x_2)+\cdots+\alpha_nf(x_n),$$

with equality if and only if $x_1=x_2=\cdots=x_n$.

In the case when $f(x)$ is strictly concave on I, the inequality is reversed. □

The convexity/concavity of a function $f(x)$ may be checked using derivative tests:

(1) Let $f(x)$ be a differentiable function on an interval I. Then $f(x)$ is strictly convex (concave) on I, if and only if $f'(x)$ is strictly increasing (decreasing) on I.

(2) Let $f(x)$ be a twice differentiable function on an interval I. Then $f(x)$ is strictly convex (concave) on I, if and only if $f''(x)>0$ ($f''(x)<0$) for all x in the interior of I.

Example 19. (same as Example 7 above) Let n be an integer greater than 1 and let $a_1,a_2,...,a_n$ be positive real numbers with the sum 1. Show that $\frac{a_1}{\sqrt{1-a_1}}+\frac{a_2}{\sqrt{1-a_2}}+\cdots+\frac{a_n}{\sqrt{1-a_n}}\ge\sqrt{\frac{n}{n-1}}$.

Solution. Since all a_i in question are from the interval $I=(0,1)$ we may consider the function $f(x)=\frac{x}{\sqrt{1-x}}$ which is differentiable on I. Calculation gives $f''(x)=\frac{4-x}{4(1-x)^{5/2}}$. It is then obvious that $f''(x)$ is positive on I, and therefore strictly convex there.

Hence we may use Jensen's inequality: $\frac{a_1}{\sqrt{1-a_1}}+\frac{a_2}{\sqrt{1-a_2}}+\cdots+\frac{a_n}{\sqrt{1-a_n}}$ $=n\cdot\frac{f(a_1)+f(a_2)+\cdots+f(a_n)}{n}\ge n\cdot f\big(\frac{a_1+a_2+\cdots+a_n}{n}\big)=n\cdot f\big(\frac{1}{n}\big)=\sqrt{\frac{n}{n-1}}$. □

Example 20. (Korea, 1998) For positive real numbers a,b,c with

$a+b+c=abc$, show that

$$\frac{1}{\sqrt{1+a^2}}+\frac{1}{\sqrt{1+b^2}}+\frac{1}{\sqrt{1+c^2}}\le\frac{3}{2}$$

and determine when equality occurs.

Solution. The formulas on the left-hand side suggest the substitution $\alpha=\arctan a$, $\beta=\arctan b$ and $\gamma=\arctan c$. This substitution together with the condition $\tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\tan\beta\tan\gamma$ implie that $0<\alpha,\beta,\gamma<\frac{\pi}{2}$ and $\alpha+\beta+\gamma=\pi$. We want to show that $\cos\alpha+\cos\beta+\cos\gamma\le\dfrac{3}{2}$.

Taking $f(x)=\cos x$ we notice that $f''(x)=-\cos x<0$ for $0<x<\pi$, hence $f(x)$ is strictly concave inside the interval $(0,\frac{\pi}{2})$.

Now we can use Jensen's inequality: $\dfrac{1}{3}(\cos\alpha+\cos\beta+\cos\gamma)=\dfrac{1}{3}\big(f(\alpha)+f(\beta)+f(\gamma)\big)\le f\big(\dfrac{\alpha+\beta+\gamma}{3}\big)=\cos\dfrac{\pi}{3}=\dfrac{1}{2}$. □

Example 21. (USA, 1974) For positive real numbers a,b,c prove that

$$a^ab^bc^c\ge(abc)^{(a+b+c)/3}.$$

Solution. Consider the function $f(x)=\ln x^x=x\ln x$. We have $f''(x)=\dfrac{1}{x}$ which is positive for $x>0$. Hence $f(x)$ is strictly convex for $x>0$.

Now, applying Jensen's inequality we get

$$\frac{1}{3}\ln(a^ab^bc^c)=\frac{\ln a^a+\ln b^b+\ln c^c}{3}=\frac{\ln a^a+\ln b^b+\ln c^c}{3}=\frac{f(a)+f(b)+f(c)}{3}$$
$$\ge f\big(\frac{a+b+c}{3}\big)=\ln\Big(\frac{a+b+c}{3}\Big)^{(a+b+c)/3}.$$

From the AM-GM inequality we have $\dfrac{a+b+c}{3}\ge\big(abc\big)^{1/3}$. Hence, $\Big(\dfrac{a+b+c}{3}\Big)^{(a+b+c)/3}\ge\big(abc\big)^{(a+b+c)/9}$ and, since logarithm is an increasing function, then we get $\ln\Big(\dfrac{a+b+c}{3}\Big)^{(a+b+c)/3}\ge\ln\big(abc\big)^{(a+b+c)/9}=\dfrac{1}{3}\ln\big(abc\big)^{(a+b+c)/3}$.

From these two results the desired inequality follows. □

8. Power Mean inequality

Let $a_1,a_2,...,a_n$ be nonnegative real numbers, k and m positive real numbers with $k\le m$. Then

$$\Big(\frac{1}{n}(a_1^k+a_2^k+\cdots a_n^k)\Big)^{1/k}\le\Big(\frac{1}{n}(a_1^m+a_2^m+\cdots a_n^m)\Big)^{1/m},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$. $\square$

Example 22. (North African MO, 1986) Let a_1, a_2, a_3 be positive real numbers. Prove that $3\big(\frac{1}{a_1a_2} + \frac{1}{a_2a_3} + \frac{1}{a_3a_1}\big) \geq 4\big(\frac{1}{a_1+a_2} + \frac{1}{a_2+a_3} + \frac{1}{a_3+a_1}\big)^2$.

Solution. The AM-GM inequality $\frac{a_1+a_2}{2} \geq \sqrt{a_1a_2}$ implies $\frac{4}{(a_1+a_2)^2} \leq \frac{1}{a_1a_2}$. Similar inequalities for the other two terms of the left-hand-side give $3\big(\frac{1}{a_1a_2} + \frac{1}{a_2a_3} + \frac{1}{a_3a_1}\big) \geq 12\big(\frac{1}{(a_1+a_2)^2} + \frac{1}{(a_2+a_3)^2} + \frac{1}{(a_3+a_1)^2}\big)$.

The rest follows from the Power Mean inequality ($n = 3$, $m = 2$ and $k = 1$): $12\big(\frac{1}{(a_1+a_2)^2} + \frac{1}{(a_2+a_3)^2} + \frac{1}{(a_3+a_1)^2}\big) = 36\big(\frac{1}{(a_1+a_2)^2} + \frac{1}{(a_2+a_3)^2} + \frac{1}{(a_3+a_1)^2}\big)/3 \geq 36\Big(\big(\frac{1}{a_1+a_2} + \frac{1}{a_2+a_3} + \frac{1}{a_3+a_1}\big)/3\Big)^2 = 4\big(\frac{1}{a_1+a_2} + \frac{1}{a_2+a_3} + \frac{1}{a_3+a_1}\big)^2$. $\square$

Example 23. (Czech Republic and Slovakia, 2000) Show that

$$\sqrt[3]{\frac{a}{b}} + \sqrt[3]{\frac{b}{a}} \leq \sqrt[3]{2(a+b)\big(\frac{1}{a} + \frac{1}{b}\big)}$$

for all positive real numbers a and b, and determine when equality occurs.

Solution. By the Power Mean inequality (with $a_1 = \big(\frac{a}{b}\big)^{1/6}$ and $a_2 = \big(\frac{b}{a}\big)^{1/6}$), we have $\Big(\frac{\sqrt[3]{\frac{a}{b}} + \sqrt[3]{\frac{b}{a}}}{2}\Big)^3 \leq \Big(\frac{\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}}{2}\Big)^2$, with equality if and only if $a/b = b/a$, or equivalently $a = b$. The desired result follows from the identity $\Big(\frac{\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}}{2}\Big)^2 = \frac{a+b}{4}\big(\frac{1}{a} + \frac{1}{b}\big)$. $\square$

Example 24. Let x, y, z be positive real numbers. Show that

$$x^5 + y^5 + z^5 \leq \frac{x^6}{\sqrt{yz}} + \frac{y^6}{\sqrt{xz}} + \frac{z^6}{\sqrt{xy}}.$$

Solution. Letting $a = \sqrt{x}$, $b = \sqrt{y}$, $c = \sqrt{z}$ and clearing the denominators we get the equivalent inequality $(a^{10} + b^{10} + c^{10})abc \leq a^{13} + b^{13} + c^{13}$. Now,

$$a^{13}+b^{13}+c^{13} = 3\Big(\sqrt[13]{\frac{a^{13}+b^{13}+c^{13}}{3}}\Big)^{13} = 3\Big(\sqrt[13]{\frac{a^{13}+b^{13}+c^{13}}{3}}\Big)^{10}\Big(\sqrt[13]{\frac{a^{13}+b^{13}+c^{13}}{3}}\Big)^{3} \geq$$

$$3\Big(\sqrt[10]{\frac{a^{10}+b^{10}+c^{10}}{3}}\Big)^{10}\Big(\frac{a+b+c}{3}\Big)^3 \geq (a^{10}+b^{10}+c^{10})\Big(\sqrt[3]{abc}\Big)^3 = (a^{10}+b^{10}+c^{10})abc. \square$$

9. Schur's inequality

Let x, y and z be nonnegative real numbers. Then for any $r > 0$,

$$x^r(x-y)(x-z) + y^r(y-z)(y-x) + z^r(z-x)(z-y) \geq 0,$$

with equality if and only if $x = y = z$ or if two of x, y, z are equal and the third is 0. □

In the case of the $r = 1$ this inequality is often written equivalently as $x^3 + y^3 + z^3 + 3xyz \geq x^2y + xy^2 + y^2z + yz^2 + z^2x + zx^2$.

In the next example we use another trick, **homogenization**, which is very useful when dealing with polynomial inequalities, provided there is an additional constraint, like $x + y + z = 1$ or $xyz = 1$. One may then multiply some terms of the inequality with the constraint in order to get all terms of the same degree.

For example, the inequality $x^2y+xz \leq 2z+7$ with $xyz = 1$ (for positive real numbers x, y, z), is equivalent to the homogeneous inequality $x^2y+xz(xyz)^{1/3} \leq 2z(xyz)^{2/3}+7(xyz)$, where each term now has degree 3.

Now, applying the substitution $x = u^3$, $y = v^3$ and $z = w^3$ and dividing by common factors we end up with the equivalent inequality $u^4v^2 + u^2w^4 \leq 2vw^5 + 7uv^2w^3$ which may be easier to deal with.

Example 25. (IMO, 1984) Let x, y, z be non negative real numbers such that $x + y + z = 1$. Prove that

$$0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27}.$$

Solution. Using the condition $x + y + z = 1$, we can reduce the given inequality to a homogeneous one, $0 \leq (xy + yz + zx)(x + y + z) - 2xyz \leq \frac{7}{27}(x + y + z)^3$. The first inequality reduces to $0 \leq xyz + x^2y + xy^2 + y^2z + yz^2 + z^2x + zx^2$, which is obviously true.

The second inequality simplifies to $7(x^3 + y^3 + z^3) + 15xyz \geq 6(x^2y + xy^2 + y^2z + yz^2 + z^2x + zx^2)$. From the AM-GM inequality we can deduce that $x^3 + y^3 + z^3 \geq 3xyz$. Hence, using Schur's inequality for $r = 1$, $7(x^3 + y^3 + z^3) + 15xyz \geq 6(x^3 + y^3 + z^3) + 18xyz \geq 6(x^2y + xy^2 + y^2z + yz^2 + z^2x + zx^2)$. □

Example 26. (IMO, 2000) Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

Solution. The expression is equivalent to the following homogeneous inequality: $\left(a-(abc)^{1/3}+\frac{(abc)^{2/3}}{b}\right)\left(b - (abc)^{1/3} + \frac{(abc)^{2/3}}{c}\right)\left(c - (abc)^{1/3} + \frac{(abc)^{2/3}}{a}\right) \leq abc.$

After the substitution $a = x^3$, $b = y^3$, $c = z^3$, $(x, y, z > 0)$, the inequality becomes $(x^2y - y^2z + z^2x)(y^2z - z^2x + x^2y)(z^2x - x^2y + y^2z) \leq x^3y^3z^3$.

Again, using the substitution $x^2y = u$, $y^2z = v$, $z^2x = w$, the inequality can be written as $3uvw + (u^3 + v^3 + w^3) \geq u^2v + v^2u + v^2w + w^2v + w^2u + u^2w$. And this exactly is what Schur's inequality for $r = 1$ says. $\square$

10. Maclaurin's inequality

Let $a_1, a_2, ..., a_n$ be positive real numbers. Then,

$$\left[\frac{1}{\binom{n}{1}}\sum_i a_i\right]^1 \geq \left[\frac{1}{\binom{n}{2}}\sum_{i<j} a_ia_j\right]^{1/2} \geq \left[\frac{1}{\binom{n}{3}}\sum_{i<j<k} a_ia_ja_k\right]^{1/3} \geq \cdots \geq$$

$$\left[\frac{1}{\binom{n}{n}}a_1a_2\cdots a_n\right]^{1/n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$. $\square$

This is a very elegant generalization of the AM-GM inequality. With the notation $S_k =$ the sum of all products of k-subsets of $\{a_1, a_2, ..., a_n\}$ divided by $\binom{n}{k}$ (for $k = 1, 2, ..., n$) we can write the Maclaurin's inequality as

$$AM = S_1 \geq S_2^{1/2} \geq S_3^{1/3} \geq \cdots \geq S_n^{1/n} = GM.$$

Example 27. (Poland, 1989) Suppose a, b, c, d are positive real numbers. Show that

$$\sqrt{\frac{ab + ac + ad + bc + bd + cd}{6}} \geq \sqrt[3]{\frac{abc + abd + acd + bcd}{4}}.$$

Solution. This is only a special case of Maclaurin's inequality: the second and the third expression for $n = 4$. $\square$

Example 28. Let a, b, c be positive real numbers. Show that

$$\frac{a^8 + b^8 + c^8}{a^3b^3c^3} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Solution. Although it is a rather easy exercise when using the AM-GM-HM inequality we will thy it bu more sofisticated method. By the Power Mean inequality $\frac{a^8 + b^8 + c^8}{3} \geq \left(\frac{a + b + c}{3}\right)^8$.
Now, using Maclaurin's inequality, $\left(\frac{a + b + c}{3}\right)^8 = \left(\frac{a + b + c}{3}\right)^6\left(\frac{a + b + c}{3}\right)^2 \geq (abc)^2 \cdot \frac{ab + bc + ca}{3}$.
Hence, $\frac{a^8 + b^8 + c^8}{a^3b^3c^3} \geq \frac{3}{(abc)^3}\left(\frac{a + b + c}{3}\right)^8 \geq \frac{ab + bc + ac}{abc} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. $\square$

11. Muirhead's inequality

In order to simplify the notation, let us introduce the symmetric summation symbol $\sum_{sym}$. Let $P(x, y, z)$ be a function of three variables, x, y and z. Let us define $\sum_{sym} P(x, y, z) = P(x, y, z) + P(x, z, y) + P(y, x, z) + P(y, z, x) + P(z, x, y) + P(z, y, x)$.

For example, for $P_1(x, y, z) = x^3,\ P_2(x, y, z) = x^2y^2z^2,\ P_3(x, y, z) = x^3y^2$, we have $\sum_{sym} P_1(x, y, z) = 2x^3 + 2y^3 + 2z^3$, $\sum_{sym} P_2(x, y, z) = 6x^2y^2z^2$ and $\sum_{sym} P_3(x, y, z) = x^3y^2 + x^2y^3 + x^3z^2 + x^2z^3 + y^3z^2 + y^2z^3$.

This notation may of course be generalized to any number $n \geq 1$ variables, but for our purposes $n = 3$ will be sufficient.

Muirhead's inequality for three variables may be now stated as follows:
Given six non negative real numbers $a_1,\ a_2,\ a_3,\ b_1,\ b_2,\ b_3$ such that
$a_1 \geq a_2 \geq a_3,\quad b_1 \geq b_2 \geq b_3,$
$a_1 \geq b_1,\ a_1 + a_2 \geq b_1 + b_2$ and $a_1 + a_2 + a_3 = b_1 + b_2 + b_3$
then, for all non negative real numbers $x,\ y,\ z$

$$\sum_{sym} x^{a_1}y^{a_2}z^{a_3} \geq \sum_{sym} x^{b_1}y^{b_2}z^{b_3}.$$

□

This particular inequality may turn out to be very useful when many other solving methods fail. Using it requires however that we deal with a homogeneous inequality.

Example 29. (USA, 1997) Prove that, for all positive real numbers a, b, c,

$$\frac{1}{a^3+b^3+abc} + \frac{1}{b^3+c^3+abc} + \frac{1}{c^3+a^3+abc} \leq \frac{1}{abc}.$$

Solution. Clearing the denominators gives a rather uninviting expression $\big((a^3 + b^3 + abc)(b^3 + c^3 + abc) + (a^3 + b^3 + abc)(c^3 + a^3 + abc) + (b^3 + c^3 + abc)(c^3 + a^3 + abc)\big)abc \leq (a^3 + b^3 + abc)(b^3 + c^3 + abc)(c^3 + a^3 + abc)$. The trick is however to multiply both sides with 2, because we may then see the left-hand side as a symmetric sum and write the inequality as

$$\sum_{sym}(a^3 + b^3 + abc)(a^3 + c^3 + abc)abc \leq 2(a^3 + b^3 + abc)(b^3 + c^3 + abc)(c^3 + a^3 + abc).$$

This may be simplified to

$$\sum_{sym}(a^7bc + 3a^4b^4c + 4a^5b^2c^2 + a^3b^3c^3) \leq \sum_{sym}(a^7bc + 2a^6b^3 + 2a^5b^2c^2 + 3a^4b^4c + a^3b^3c^3).$$

The last inequality reduces to $\sum_{sym}\left(a^6b^3 - a^5b^2c^2\right) \geq 0$, which is clearly true by Muirhead's inequality (with $a_1 = 6, a_2 = 3, a_3 = 0, b_1 = 5, b_2 = 2, b_3 = 2$). $\square$

Example 30. (IMO, 1995) Prove the following inequality for positive real numbers x, y, z, such that $xyz = 1$:

$$\frac{1}{x^3(y+z)} + \frac{1}{y^3(z+x)} + \frac{1}{z^3(x+y)} \geq \frac{3}{2}.$$

Solution. Suppose that we find no other "clever" way of solving the problem and the only methods we can apply is by direct counting.

We start with making the inequality homogenous by multiplying the denominator of the right-hand side by $(xyz)^{4/3}$. Then, in order to make our calculations easier we make the substitution $x = a^3$, $y = b^3$ and $z = c^3$, with $a, b, c > 0$. The inequality is then equivalent to

$$\frac{1}{a^9(b^3+c^3)} + \frac{1}{b^9(c^3+a^3)} + \frac{1}{c^9(a^3+b^3)} \geq \frac{3}{2a^4b^4c^4}.$$

Clearing the denominators leads to the following inequality (it should be mentioned that you don't have to first multiply all terms in order to find the appropriate symmetric sum. After practising with some exercises you will be able to quickly discover the pattern and find those sums.)

$$\sum_{sym}(a^{12}b^{12} + 2a^{12}b^9c^3 + a^9b^9c^6) \geq \sum_{sym}(3a^{11}b^8c^5 + a^8b^8c^8).$$

This may in turn be reduced to

$$\sum_{sym}\left(a^{12}b^{12} - a^{11}b^8c^5\right) + 2\sum_{sym}\left(a^{12}b^9c^3 - a^{11}b^8c^5\right) + \sum_{sym}\left(a^9b^9c^6 - a^8b^8c^8\right) \geq 0.$$

The Muirhead's inequality implies now that every term on the left-hand side is ≥ 0.

There are of course many other ways to solve this problem. One is to use the Cauchy-Schwarz inequality. We may substitute $x = \frac{1}{a}$, $y = \frac{1}{b}$ and $z = \frac{1}{c}$, and then the inequality becomes $\frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y} \geq \frac{3}{2}$.

Now, since $x + y + z = \frac{x}{\sqrt{y+z}} \cdot \sqrt{y+z} + \frac{y}{\sqrt{x+z}} \cdot \sqrt{x+z} + \frac{z}{\sqrt{x+y}} \cdot \sqrt{x+y}$, then, by the Cauchy-Schwarz inequality $(x+y+z)^2 \leq \left(\frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y}\right)\left((y+z)+(x+z)+(x+y)\right)$.

Hence, $\frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y} \geq \frac{x+y+z}{2} \geq \frac{3}{2}\sqrt[3]{xyz} = \frac{3}{2}$. The last inequality follows from the AM-GM inequality.

Yet another way to approach the problem is by using the Rearrangement inequality. Because the inequality we want to show is symmetric, and using the same substitution as in the previous solution, we may assume that $x \geq y \geq z$. Then $x^2 \geq y^2 \geq z^2$ and $\frac{1}{y+z} \geq \frac{1}{x+z} \geq \frac{1}{x+y}$. Now, since

$$\begin{bmatrix} x^2 & y^2 & z^2 \\ \frac{1}{y+z} & \frac{1}{x+z} & \frac{1}{x+y} \end{bmatrix} \geq \begin{bmatrix} x^2 & y^2 & z^2 \\ \frac{1}{x+y} & \frac{1}{y+z} & \frac{1}{x+z} \end{bmatrix} \text{ and } \begin{bmatrix} x^2 & y^2 & z^2 \\ \frac{1}{y+z} & \frac{1}{x+z} & \frac{1}{x+y} \end{bmatrix} \geq \begin{bmatrix} x^2 & y^2 & z^2 \\ \frac{1}{x+z} & \frac{1}{x+y} & \frac{1}{y+z} \end{bmatrix}$$

then, after adding those two inequalities and dividing by 2, we get

$$\frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y} \geq \frac{1}{2}\left(\frac{x^2+y^2}{x+y} + \frac{y^2+z^2}{y+z} + \frac{x^2+z^2}{x+z}\right).$$

Since $s^2 + t^2 \geq (s+t)^2/2$ then the right hand side is $\geq \frac{1}{2}\left(\frac{x+y}{2} + \frac{y+z}{2} + \frac{x+z}{2}\right) = \frac{x+y+z}{2}$. Finally, the AM-GM inequality implies that $\frac{x+y+z}{2} \geq \frac{3}{2}\sqrt[3]{xyz} = \frac{3}{2}$. □

12. Substitutions

The process of treating inequalities may sometimes be simplified by suitable substitutions, as we already have seen before in Example 20.

Example 31. (Russia, 2000) For real numbers x, y such that $0 \leq x, y \leq 1$, prove that

$$\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} \leq \frac{2}{\sqrt{1+xy}}.$$

Solution. If $x = 0$, the problem reduces to the inequality $1 + \frac{1}{\sqrt{1+y^2}} \leq 2$, which is obviously true. So, suppose $0 < x, y \leq 1$. The inequality looks almost like a candidate for Jensen's inequality with the concave function $f(x) = \frac{1}{\sqrt{1+x^2}}$. The only obstacle is the right-hand product xy instead of the sum $x + y$.

Because of that it seems to be appropriate to instead study the function $g(s) = \frac{1}{\sqrt{1+e^{-2s}}}$. To that end we make the substitutions $x = e^{-u}$ and $y = e^{-v}$ for some nonnegative real numbers u and v. The problem then reduces to the equivalent inequality $\frac{1}{\sqrt{1+e^{-2u}}} + \frac{1}{\sqrt{1+e^{-2v}}} \leq \frac{2}{\sqrt{1+e^{-(u+v)}}}$.

To finish the solution, all we need is to show that the function $g(s)$ above is concave for $s \geq 0$. Easy calculations give $g''(s) = \frac{1-2e^{2s}}{(1+e^{-2s})^{5/2}e^{4s}}$. The denominator is certainly positive, while the numerator is negative, because $e^{2s} \geq 1$ for $s \geq 0$. Thus, $g(s)$ is strictly concave for $s \geq 0$ and the solution is complete. □

Example 32. Prove that, for all positive real numbers a, b, c, d,

$$\sqrt{ab} + \sqrt{cd} \leq \sqrt{(a+d)(b+c)}.$$

Solution. Dividing the inequality with the expression on the right-hand side we get the equivalent inequality $\sqrt{\frac{a}{a+d} \cdot \frac{b}{b+c}} + \sqrt{\frac{c}{b+c} \cdot \frac{d}{a+d}} \leq 1$, or

$$\sqrt{\frac{a}{a+d} \cdot \frac{b}{b+c}} + \sqrt{\left(1 - \frac{b}{b+c}\right) \cdot \left(1 - \frac{a}{a+d}\right)} \leq 1$$

Letting $\frac{a}{a+d} = \sin^2 \alpha$ and $\frac{b}{b+c} = \cos^2 \beta$ for some $\alpha, \beta,\quad 0 < \alpha, \beta < \frac{\pi}{2}$, the left-hand side can be written as $\sin\alpha \sin\beta + \cos\alpha\cos\beta = \cos(\alpha - \beta)$, which obviously is ≤ 1. □

PROBLEMS 1

With each problem comes my suggestion for a strategy of solving it. Such a strategy is never unique and these problems may certainly be solved by alternative methods as well.

1. (St. Petersburg, 1997) Prove that for $x, y, z \geq 2,\ (x+y^3)(y+z^3)(z+x^3) \geq 125xyz$.
(Hint: $x + y^3 \geq x + 4y$. Now, apply the AM-GM inequality.)

2. (Russia, 1995) Prove that for $x, y > 0$, $\frac{x}{x^4+y^2} + \frac{y}{x^2+y^4} \leq \frac{1}{xy}$.
(Hint: Apply the AM-GM inequality to denominators..)

3. Let a, b, c, d be positive real numbers. Prove that $\frac{1}{a} + \frac{4}{b} + \frac{4}{c} + \frac{16}{d} \geq \frac{64}{a+b+c+d}$.
(Hint: the AM-GM inequality.)

4. (Asian-Pacific MO, 1998) Let a, b, c be positive real numbers. Prove that $\left(1 + \frac{a}{b}\right)\left(1 + \frac{b}{c}\right)\left(1 + \frac{c}{a}\right) \geq 2\left(1 + \frac{a+b+c}{\sqrt[3]{abc}}\right)$.
(Hint: the AM-GM inequality twice.)

5. (Poland, 1990) Let x, y, z be positive real numbers such that $xyz = 2$. Show that $x^2 + y^2 + z^2 + xy + yz + xz \geq 2(\sqrt{x} + \sqrt{y} + \sqrt{z})$.
(Hint: You may start by showing that $x^2 + y^2 + z^2 \geq xy + yz + zx$ and then use the AM-GM inequality.)

6. (Vietnam, 1998) Let n be an integer, $n \geq 2$, and let $x_1, x_2, ..., x_n$ be positive numbers satisfying $\dfrac{1}{x_1 + 1998} + \dfrac{1}{x_2 + 1998} + \cdots + \dfrac{1}{x_n + 1998} = \dfrac{1}{1998}$. Prove that $\dfrac{\sqrt[n]{x_1 x_2 \cdots x_n}}{n-1} \geq 1998$.
(Hint: For $y_i = \dfrac{1998}{x_i + 1998}$ observe that $1 - y_i = \sum_{k \neq i} y_k$. Now use the AM-GM inequality.)

7. (Belarus, 1999) Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that $\dfrac{1}{1+ab} + \dfrac{1}{1+bc} + \dfrac{1}{1+ca} \geq \dfrac{3}{2}$.
(Hint: use the AM-HM inequality and the inequality $a^2 + b^2 + c^2 \geq ab + bc + ca$.)

8. (St. Petersburg, 1999) Let $x_0 > x_1 > \cdots > x_n$ be real numbers. Prove that $x_0 + \dfrac{1}{x_0 - x_1} + \dfrac{1}{x_1 - x_2} + \cdots + \dfrac{1}{x_{n-1} - x_n} \geq x_n + 2n$.
(Hint: the AM-GM inequality for $t + \dfrac{1}{t} \geq 2$.)

9. Find all pairs of positive real numbers x, y such that $\dfrac{64x^2y^2}{4x^2 + y^2} = (x+1)(y+2)(2x+y)$.
(Hint: After clearing the denominator use the AM-GM inequality on the right-hand side expression.)

10. (Poland, 1990) Let a, b be two positive real numbers. Find all pairs of positive real numbers x, y such that $\dfrac{27xy}{(1+2ax)(1+2by)} = \dfrac{1}{ab} + \dfrac{x}{a} + \dfrac{y}{b}$.
(Hint: One may use the AM-GM inequality three times: for $1 + 2ax$, for $1 + 2by$ and for the right hand side of the equality.)

11. Let n be a positive integer and let a, b, c be positive real numbers. Show the inequality $\dfrac{a^{n+1} + b^{n+1} + c^{n+1}}{a^n + b^n + c^n} \geq \dfrac{a+b+c}{3}$.
(Hint: Chebyshev's inequality.)

12. Solve the problem from Example 21 by using Chebyshev's inequality.
(Hint: Write $\ln(a^a b^b c^c)$ as $a \ln a + b \ln b + c \ln c$.)

13. (IMO, 1978) Let $a_1, a_2, ..., a_n$ be distinct positive integers. Prove that

$$\frac{a_1}{1^2} + \frac{a_2}{2^2} + \cdots + \frac{a_n}{n^2} \geq \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}.$$

(Hint: The Rearrangement inequality.)

14. Let $x_1, x_2, ..., x_n$ be positive numbers. Show that $\dfrac{x_1}{x_2} + \dfrac{x_2}{x_3} + \cdots + \dfrac{x_n}{x_1} \geq n$.
(Hint: The Rearrangement inequality.)

15. Let a, b, c be positive real numbers. Use the Rearrangement inequality in order to prove that $\dfrac{a^8+b^8+c^8}{a^3b^3c^3} \ge \dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$. (Compare with Example 28 above).

16. (Korea, 2000) The real numbers a, b, c, x, y, z satisfy $a \ge b \ge c > 0$ and $x \ge y \ge z > 0$ Prove that $\dfrac{a^2x^2}{(by+cz)(bz+cy)} + \dfrac{b^2y^2}{(cz+ax)(cx+az)} + \dfrac{c^2z^2}{(ax+by)(ay+bx)} \ge \dfrac{3}{4}$.

(Hint: With the notation $\alpha = a^2x^2$, $\beta = b^2y^2$, $\gamma = c^2z^2$ and using the Rearrangement inequality, show that the expression is $\ge \dfrac{1}{2}\left(\dfrac{\alpha}{\beta+\gamma} + \dfrac{\beta}{\gamma+\alpha} + \dfrac{\gamma}{\alpha+\beta}\right)$. Now use the Cauchy-Schwarz inequality.)

17. (Ireland, 1999) Let a, b, c, d be positive real numbers whose sum is 1. Prove that $\dfrac{a^2}{a+b} + \dfrac{b^2}{b+c} + \dfrac{c^2}{c+d} + \dfrac{d^2}{d+a} \ge \dfrac{1}{2}$ with equality if and only if $a = b = c = d = 1/4$.

(Hint: Apply the Cauchy-Schwarz inequality to the left-hand side multiplied by $(a+b)+(b+c)+(c+d)+(d+a)$.)

18. (Czech Republic and Slovakia, 1999) For arbitrary positive real numbers a, b, c, prove the inequality $\dfrac{a}{b+2c} + \dfrac{b}{c+2a} + \dfrac{c}{a+2b} \ge 1$.

(Hint: Apply the Cauchy-Schwarz inequality to the left-hand side multiplied by $a(b+2c)+b(c+2a)+c(a+2b)$ or substitute $x = b+2c$, $y = c+2a$, $z = a+2b$ and use AM-GM.)

19. Let a, b, c, d be positive real numbers such that $c^2+d^2 = (a^2+b^2)^3$. Show that $\dfrac{a^3}{c} + \dfrac{b^3}{d} \ge 1$.

(Hint: Using the Cauchy-Schwarz inequality show that $(a^3/c + b^3/d)(ac+bd) \ge (a^2+b^2)^2 \ge ac+bd$.)

20. For non negative real numbers $x_1, x_2, ..., x_n$ show that
$(1+x_1)(1+x_2)\cdots(1+x_n) \ge \left(1+\sqrt[n]{x_1x_2\cdots x_n}\right)^n$.

(Hint: Let $a_i = \sqrt[n]{x_i}$ and then apply Hölder's inequality.)

21. Use Hölder's inequality to prove that $a^3+b^3+c^3 \ge a^2b+b^2c+c^2a$ for all positive real numbers a, b, c.

22. Given a, b, c, p, q, r, positive real numbers with $p+q+r = 1$, prove that $a+b+c \ge a^pb^qc^r + a^qb^rc^p + a^rb^pc^q$.

(Hint: Hölder's inequality.)

23. Let $x_k \in [1, 2]$, $k = 1, 2, ..., n$. Prove that $2\left(\sum_{k=1}^{n} x_k\right) \cdot \left(\sum_{k=1}^{n} \dfrac{1}{x_k}\right)^2 \ge n^3$.

(Hint: Hölder's inequality with $p = \frac{1}{3}, q = \frac{2}{3}$.)

24. Let a, b, c, x, y, x be positive real numbers. Prove that $\sqrt{\frac{ax^2+by^2+cz^2}{a+b+c}}$ $+\sqrt{\frac{ay^2+bz^2+cx^2}{a+b+c}}+\sqrt{\frac{az^2+bx^2+cy^2}{a+b+c}} \geq x+y+z.$
(Hint: Minkowski's inequality.)

25. Let a, b, c, d be positive real numbers. Prove that $\frac{1}{a}+\frac{1}{b}+\frac{4}{c}+\frac{16}{d} \geq \frac{64}{a+b+c+d}$.
(Observe the improvement compared with the problem 3 above. Hint: Hölder's inequality.)

26. (Ireland, 1998) Let a, b, c be positive real numbers. Apply Jensen's inequality to show that $\frac{9}{a+b+c} \leq 2\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right) \leq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$

27. Let x, y, z be three positive real numbers such that $x+y+z=1$. Prove that $\frac{x}{x+1}+\frac{y}{y+1}+\frac{z}{z+1} \leq \frac{3}{4}.$
(Hint: Jensen's or the AM-GM inequality).

28. Find the minimum value of the expression $\sum_{k=1}^{n} \frac{1}{4a_k-a_k^3}$, where $0<a_1, a_2, ..., a_n<2$ and $\sum_{k=1}^{n} a_k = n.$
(Hint: Jensen's inequality or the AM-GM and Hölder's inequalities.)

29. (Nordic, 1992) Show that among all triangles with the inscribed circle of radius 1, the equilateral triangle has the minimum circumference.
(Hint: Letting 2α, 2β, 2γ be angles of the triangle, half the circumference may be expressed as $f(\alpha)+f(\beta)+f(\gamma)$, where $f(x)=\frac{1}{\tan x}$, for $0<x<\pi/2$.)

30. Let x, y, z be three real numbers such that $0<x, y, z<4$ and $xyz=1$. Prove that $\sqrt{\frac{x}{x+8}}+\sqrt{\frac{y}{y+8}}+\sqrt{\frac{z}{z+8}} \geq 1.$
(Hint: Looks like a candidate for Jensen's inequality. However, in this problem we have a product condition $xyz=1$, where we instead would like to have the sum condition $x+y+z=constant$. This obstacle can however be removed by introducing new variables a, b, c by substitutions $x=e^a$, $y=e^b$ and $z=e^c$. Now, study the function $f(s)=\left(\frac{e^s}{e^s+8}\right)^{1/2}$. This is in fact a part of a problem from IMO 2001. For the whole problem you must remove the constraint $x, y, z<4$. If only one of the variables is ≥ 4 you can still use Jensen's inequality (for those two terms with variable values less than 4). If at least two of variables are greater than 4 the proof is straightforward.)

31. (Russia, 1999) Let x and y bet two positive real numbers such that $x^3+y^3>2$. Prove that

$x^2 + y^3 < x^3 + y^4$.

(Hint: You may start with the Power Mean inequality for $(x^2 + y^2)/2$ and $(x^3 + y^3)/2$.

32. (Iran, 1998) Let a_1, a_2, a_3, a_4 be positive real numbers such that $a_1a_2a_3a_4 = 1$. Prove that $\sum_{i=1}^{4} a_i^3 \geq \max\Big\{\sum_{i=1}^{4} a_i, \sum_{i=1}^{4} \frac{1}{a_i}\Big\}$.

(Hint: For the first inequality you may use the Power Mean inequality. The second follows from the AM-GM inequality.)

33. Let a, b, c, d be positive real numbers. Prove that $\frac{a^3 + b^3 + c^3}{a + b + c} + \frac{a^3 + b^3 + d^3}{a + b + d} + \frac{a^3 + c^3 + d^3}{a + c + d} + \frac{b^3 + c^3 + d^3}{b + c + d} \geq a^2 + b^2 + c^2 + d^2$.

(Hint: Using the Power Mean inequality show that $a^3 + b^3 + c^3 \geq \frac{1}{3}(a + b + c)(a^2 + b^2 + c^2)$.)

34. (Poland, 1999) Let x, y, z be three positive real numbers such that $x + y + z = 1$. Prove that $x^2 + y^2 + z^2 + 2\sqrt{3xyz} \leq 1$.

(Hint: Make the inequality homogeneous.)

35. (UK, 1999) Some three nonnegative real numbers p, q, r satisfy $p + q + r = 1$. Prove that $7(pq + qr + rp) \leq 2 + 9pqr$.

(Hint: Use the condition $p + q + r = 1$ to make the inequality homogeneous and then use Schur's inequality.)

36. Let $a_1, a_2, ..., a_n$ be positive real numbers such that $(1 + a_1)(1 + a_2)\cdots(1 + a_n) = 2^n$. Prove that $a_1a_2\cdots a_n \leq 1$.

(Hint: With the notation of the Maclaurin's inequality write $(1 + a_1)(1 + a_2)\cdots(1 + a_n)$ as a sum of S_1, ..., S_n and then show that this sum is $\geq (1 + S_n^{1/n})^n$. The binomial theorem may be useful here. Or you may use the same metod as in problem 20.)

37. (Asian-Pacific MO, 1998) Let a, b, c be positive real numbers. Prove that $\big(1 + \frac{a}{b}\big)\big(1 + \frac{b}{c}\big)\big(1 + \frac{c}{a}\big) \geq 2\big(1 + \frac{a+b+c}{\sqrt[3]{abc}}\big)$.

(Hint: Make the substitution $a = x^3$ and so on, then apply Muirhead's inequality.)

38. Prove the inequality $\frac{x}{(x + y)(x + z)} + \frac{y}{(y + z)(y + x)} + \frac{z}{(z + x)(z + y)} \leq \frac{9}{4(x + y + z)}$ for all positive real numbers x, y, z.

(Hint: Use the Muirhead's inequality.)

39. For real numbers $-1 \leq x, y \leq 1$, find the maximum value of the function $f(x, y) = xy + x\sqrt{1 - y^2} + y\sqrt{1 - x^2} - \sqrt{(1 - x^2)(1 - y^2)}$.

(Hint: The expression invites to the trigonometric substitution $x = \cos\alpha$, $y = \cos\beta$, for $0 \leq \alpha, \beta \leq \pi$.)

40. Let x, y, z be positive real numbers such that $xy + yz + zx = 1$. Prove the following inequality $\frac{2x(1-x^2)}{(1+x^2)^2} + \frac{2y(1-y^2)}{(1+y^2)^2} + \frac{2z(1-z^2)}{(1+z^2)^2} \leq \frac{x}{1+x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2}$.

(Hint: Since the expressions are similar to those of $\sin\alpha = \frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$ and $\cos\alpha = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$, you can try to use the substitution $x = \tan\frac{\alpha}{2}$, $y = \tan\frac{\beta}{2}$, $z = \tan\frac{\gamma}{2}$.)

PROOFS OF THE INEQUALITIES

1. AM-GM-HM inequality

Considering the importance and usefulness of the AM-GM inequality, we give here three different proofs of it. Yet another proof can be found in the last section of this text.

Proof 1 (by induction): We start by proving the inequality for $n = 2$.

We have $(\sqrt{a_i} - \sqrt{a_2})^2 \geq 0$. We develop the left-hand side of this inequality, move the negative term over to the right-hand side and then divide both sides by 2 to get $\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2}$. Thus the AM-GM inequality holds for $n = 2$.

Now assume that the inequality holds for some $n \geq 2$. Then we will show that it holds for $2n$.

We have $\sum_{i=1}^{2n} a_i = \sum_{i=1}^{n} a_i + \sum_{i=n+1}^{2n} a_i \geq n\left(\sqrt[n]{\prod_{i=1}^{n} a_i} + \sqrt[n]{\prod_{i=n+1}^{2n} a_i}\right) \geq$ $2n\sqrt{\sqrt[n]{\prod_{i=1}^{n} a_i} \cdot \sqrt[n]{\prod_{i=n+1}^{2n} a_i}} = 2n\sqrt[2n]{\prod_{i=1}^{2n} a_i}$.

The first inequality in the expression above holds according to our assumption. The second inequality follows from the AM-GM inequality for $n = 2$.

Thus, by the induction argument the AM-GM inequality holds for $n = 2^k$, ($k = 1, 2, \ldots$).

At last we assume that the inequality holds for some $n \geq 2$. Then we will show that it holds for $n - 1$.

Let $a_1, a_2, \ldots, a_{n-1}$ be positive real numbers and let $a_n = g = \sqrt[n-1]{\prod_{i=1}^{n-1} a_i}$. Then we have

$\sum_{i=1}^{n-1} a_i + g \geq n \sqrt[n]{\prod_{i=1}^{n-1} a_i \cdot g} = n \sqrt[n]{g^{n-1} \cdot g} = ng$. Hence we have

$\sum_{i=1}^{n-1} a_i + \sqrt[n-1]{\prod_{i=1}^{n-1} a_i} \geq n \sqrt[n-1]{\prod_{i=1}^{n-1} a_i}$, i.e. $\sum_{i=1}^{n-1} a_i \geq (n-1) \sqrt[n-1]{\prod_{i=1}^{n-1} a_i}$.

The inequality in the expression above holds according to our assumption.

Thus the AM-GM inequality holds for all $n \geq 2$. □

Proof 2 (by easy analysis):

First we prove one useful theorem, which in fact is a generalization of the AM-GM inequality (the weighted version).

Theorem 1: Let $a_1, a_2, ..., a_n$ be positive real numbers. Let then $\alpha_1, \alpha_2, ..., \alpha_n$ be positive real numbers such that $\alpha_1 + \alpha_2 + ... + \alpha_n = 1$. We set $G = \prod_{k=1}^{n} a_k^{\alpha_k}$ and $A = \sum_{k=1}^{n} \alpha_k a_k$. Then we have $G \leq A$ with equality if and only if $a_1 = a_2 = ... = a_n$.

Proof: We set $a_k = (1 + x_k)A$. We note that $x_k > -1$, $(k = 1, 2, ..., n)$ and that $\sum_{k=1}^{n} \alpha_k x_k = 0$.

Hence $G = \prod_{k=1}^{n} a_k^{\alpha_k} = \prod_{k=1}^{n} ((1 + x_k)A)^{\alpha_k} = A \prod_{k=1}^{n} (1 + x_k)^{\alpha_k} \leq A \prod_{k=1}^{n} e^{x_k \alpha_k} = A$.

One realizes that the inequality $A \prod_{k=1}^{n} (1 + x_k)^{\alpha_k} \leq A \prod_{k=1}^{n} e^{x_k \alpha_k}$ holds by studying the function $f(x) = e^x - (1 + x)$. (We have $f(0) = 0$, and since $f'(x) = e^x - 1$, we have $f'(x) = 0$ for $x = 0$. Finally we have $f''(0) = 1$, which means that $f(x)$ has minimum at $x = 0$. Thus, $f(x) \geq 0$ for all $x \in \mathbb{R}$). We get equality only when $x_k = 0$, $(k = 1, 2, ..., n)$, i.e. when $a_1 = a_2 = ... = a_n$. □

Now, if we in Theorem 1 set $\alpha_k = \frac{1}{n}$, $(k = 1, 2, ..., n)$, we get
$\sqrt[n]{a_1 \cdot a_2 \cdot ... + \cdot a_n} \leq \frac{a_1 + a_2 + ... + a_n}{n}$, which is the AM-GM inequality. □

Proof 3: The AM-GM inequality also follows easily from Jensen's inequality.

Let $a_1, a_2, ..., a_n$ be positive real numbers. We note that the function $f(x) = e^x$ is strictly convex, and that $n \cdot \frac{1}{n} = 1$. Jensen's inequality (with $x_i = \ln a_i$, $i = 1, 2, ..., n$) then yields

$$e^{\left(\frac{1}{n} \ln a_1 + \frac{1}{n} \ln a_2 + ... + \frac{1}{n} \ln a_n\right)} \leq \frac{1}{n} e^{\ln a_1} + \frac{1}{n} e^{\ln a_2} + ... + \frac{1}{n} e^{\ln a_n},$$

which after simplifying becomes the AM-GM inequality. □

It may also be worth pointing out that the AM-GM inequality follows directly from the Power Mean inequality. (Just keep the first and the last part of that inequality and you get the AM-GM inequality.)

Proof of the GM-HM inequality: Follows easily from the AM-GM inequality:

Let $a_1, a_2, ..., a_n$ be positive real numbers. Then $\frac{1}{a_1}, \frac{1}{a_2}, ..., \frac{1}{a_n}$ are also positive real numbers. According to the AM-GM inequality, we have $\sqrt[n]{\frac{1}{a_1} \cdot \frac{1}{a_2} \cdot ... \cdot \frac{1}{a_n}} \leq \frac{\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}}{n}$. When we invert both sides of the inequality, then, of course, the sign of the inequality gets reversed. Hence $\sqrt[n]{a_1 \cdot a_2 \cdot ... \cdot a_n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}}$. $\square$

2. Chebyshev's inequality

We note that $\sum_{i=1}^{n}\sum_{j=1}^{n}(a_ib_i - a_ib_j) = \sum_{i=1}^{n}(na_ib_i - a_i\sum_{j=1}^{n}b_j) = n\sum_{i=1}^{n}a_ib_i - \sum_{i=1}^{n}a_i\sum_{j=1}^{n}b_j = n\sum_{i=1}^{n}a_ib_i - \sum_{i=1}^{n}a_i\sum_{i=1}^{n}b_i$, and that $\sum_{i=1}^{n}\sum_{j=1}^{n}(a_jb_j - a_jb_i) = \sum_{i=1}^{n}\Big(\sum_{j=1}^{n}a_jb_j - \Big(\sum_{j=1}^{n}a_j\Big) \cdot b_i\Big) = n\sum_{j=1}^{n}a_jb_j - \sum_{j=1}^{n}a_j\sum_{i=1}^{n}b_i = n\sum_{i=1}^{n}a_ib_i - \sum_{i=1}^{n}a_i\sum_{i=1}^{n}b_i$.

It follows that $n\sum_{i=1}^{n}a_ib_i - \sum_{i=1}^{n}a_i\sum_{i=1}^{n}b_i = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_ib_i - a_ib_j) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_jb_j - a_jb_i) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_ib_i - a_ib_j + a_jb_j - a_jb_i) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}((a_i - a_j)(b_i - b_j))$.

Now we realize that $(a_i - a_j)(b_i - b_j) \geq 0$ for $i, j = 1, 2, ..., n$. (The conditions $a_i \leq a_2 \leq ... \leq a_n$, $b_i \leq b_2 \leq ... \leq b_n$ lead to the fact that the two factors on the left-hand side of the last mentioned inequality will always have the same sign.) Hence, $n\sum_{i=1}^{n}a_ib_i - \sum_{i=1}^{n}a_i\sum_{i=1}^{n}b_i \geq 0$. Moving the right term on the left-hand side of this inequality to the right-hand side and then dividing both sides by n^2 yields Chebyshev's inequality.

It is not hard to realize that in Chebyshev's inequality we will have equality if and only if $a_1 = a_2 = ... = a_n$ or $b_1 = b_2 = ... = b_n$ This follows from the facts that $(a_i - a_j)(b_i - b_j) \geq 0$, for $i, j = 1, 2, ..., n$. To realize that the inequality is reversed if we have $a_i \leq a_2 \leq ... \leq a_n$ and $b_i \geq b_2 \geq ... \geq b_n$ is no more difficult. This follows from $(a_i - a_j)(b_i - b_j) \leq 0$, for $i, j = 1, 2, ..., n$ (these inequalities are reversed because now the conditions lead to the fact that the two factors on the left-hand side will always have the opposite sign). □

Another proof: We will now show that Chebyshev's inequality easily can be derived from the Rearrangement inequality. Chebyshev's inequality can be expressed as $(a_1 + a_2 + ... + a_n)(b_1 + b_2 + ... + b_n) \leq n(a_1b_1 + a_2b_2 + .. + a_nb_n)$. On the left-hand side we multiply the two parenthesis together. We then get n^2 terms on that side. Then we arrange these terms in n groups as follows: $(a_1b_1 + a_2b_2 + ... + a_nb_n) + (a_1b_2 + a_2b_3 + ... + a_nb_1) + ... + (a_1b_n + a_2b_1 + ... + a_nb_{n-1})$. According to the Rearrangement inequality $a_1b_1 + a_2b_2 + .. + a_nb_n \geq$ anyone of these n groups of numbers, and Chebyshev's inequality follows. Using the same technique one can easily show the case when Chebyshev's inequality is reversed. □

3. Rearrangement inequality

We start by showing the inequality for $n = 2$. Let $a_1 \leq a_2$ and $b_i \leq b_2$ be real numbers. Then we have $(a_2 - a_1)(b_2 - b_1) \geq 0$, since both factors on the left-hand side of the inequality are definitely nonnegative. Multiplying these factors together and then rearranging the terms yields the inequality $a_1b_1 + a_2b_2 \geq a_1b_2 + a_2b_1$, which is the Rearrangement inequality for $n = 2$. We realize that we have equality only if $a_1 = a_2$ or $b_1 = b_2$.

Now we turn to the general case.

Let $b_1 \leq b_2 \leq ... \leq b_n$ and $c_1, c_2, ..., c_n$ be real numbers. Let then $a_1, a_2, ..., a_n$ be a permutation of $c_1, c_2, ..., c_n$ such that $a_1b_1 + a_2b_2 + ... + a_nb_n$ is maximal. Now, assume that we for some $i < j$ have $a_i > a_j$. Then we have $a_ib_j + a_jb_i \geq a_ib_i + a_jb_j$ (the case $n = 2$). Thus, $a_1b_1 + a_2b_2 + ... + a_nb_n$ is not maximal unless $a_1 \leq a_2 \leq ... \leq a_n$ or $b_i = b_j$ for all $i < j$ such that $a_i > a_j$. In the later case the numbers a_i, a_j can change places so that we get $a_1 \leq a_2 \leq ... \leq a_n$. Hence, $a_1b_1 + a_2b_2 + ... + a_nb_n$ is maximal when $a_1 \leq a_2 \leq ... \leq a_n$.

Now we note that $-(a_1b_1 + a_2b_2 + ... + a_nb_n) = (-a_1)b_1 + (-a_2)b_2 + ... + (-a_n)b_n$ is maximal when $-a_1 \leq -a_2 \leq ... \leq -a_n$ and $b_1 \leq b_2 \leq ... \leq b_n$. From that follows that $a_1b_1 + a_2b_2 + ... + a_nb_n$ is minimal under the same conditions. But the condition $-a_1 \leq -a_2 \leq ... \leq -a_n$ is equivalent to the condition $a_1 \geq a_2 \geq ... \geq a_n$. Thus, $a_1b_1 + a_2b_2 + ... + a_nb_n$ is minimal when $b_1 \leq b_2 \leq ... \leq b_n$ and $a_1 \geq a_2 \geq ... \geq a_n$. □

4. Cauchy-Schwarz inequality

Let $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ be real numbers. Let us look at the polynomial $\sum_{i=1}^{n}(a_ix + b_i)^2 = \Big(\sum_{i=1}^{n} a_i^2\Big)x^2 + 2\Big(\sum_{i=1}^{n} a_ib_i\Big)x + \sum_{i=1}^{n} b_i^2$.

Since $\sum_{i=1}^{n}(a_i x + b_i)^2 \geq 0$ for all real numbers x, it is obvious that $\Big(\sum_{i=1}^{n} a_i^2\Big)x^2 + 2\Big(\sum_{i=1}^{n} a_i b_i\Big)x + \sum_{i=1}^{n} b_i^2 \geq 0$ too. It follows that the discriminant for this polynomial is ≤ 0. The discriminant for a quadratic polynomial $ax^2 + bx + c$ is $b^2 - 4ac$. Thus, $4\Big(\sum_{i=1}^{n} a_i b_i\Big)^2 - 4\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \leq 0$. If we then move the right-hand term on the left-hand side of this inequality to the right-hand side and finally divide both sides by 4, we end up with the Cauchy-Schwarz inequality.

So, why does the discriminant for the above polynomial have to be ≤ 0? If we solve the equation $ax^2 + bx + c = 0$ for arbitrary real numbers a, b, c using quadratic completion, we get the solution $x = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}$. Now, since we have $ax^2 + bx + c \geq 0$, we know that the above equation will have no real roots or one double root. If we have no real roots, the sum of the terms under the root sign will be negative, i.e. $b^2 - 4ac < 0$. If we have a double root, we will have $b^2 - 4ac = 0$.

It follows from the fact that $\sum_{i=1}^{n}(a_i x + b_i)^2 \geq 0$ for all real numbers x, that the Cauchy-Schwarz inequality will have equality if and only if the number sequences a and b are proportional. $\square$

Another proof of this inequality is supplied in the last part of this text.

5. Hölder's inequality

Set $p = \frac{1}{s}$ and $q = \frac{1}{t}$. Then Hölder's inequality becomes $\sum_{k=1}^{n} a_k b_k \leq \Big(\sum_{k=1}^{n} a_k^s\Big)^{\frac{1}{s}} \cdot \Big(\sum_{k=1}^{n} b_k^t\Big)^{\frac{1}{t}}$. The inequality can then be expressed in the following way: $\sum_{k=1}^{n}\Big(\frac{a_k^s}{\sum_{i=1}^{n} a_i^s}\Big)^{\frac{1}{s}} \cdot \Big(\frac{b_k^t}{\sum_{i=1}^{n} b_i^t}\Big)^{\frac{1}{t}} \leq 1$. We note that $\frac{1}{s} + \frac{1}{t} = 1$. Then, applying Theorem 1 (see proof of the AM-GM inequality) to the left-hand side of this inequality yields

$$\sum_{k=1}^{n}\Big(\frac{a_k^s}{\sum_{i=1}^{n} a_i^s}\Big)^{\frac{1}{s}} \cdot \Big(\frac{b_k^t}{\sum_{i=1}^{n} b_i^t}\Big)^{\frac{1}{t}} \leq \sum_{k=1}^{n}\Big(\frac{1}{s}\frac{a_k^s}{\sum_{i=1}^{n} a_i^s} + \frac{1}{t}\frac{b_k^t}{\sum_{i=1}^{n} b_i^t}\Big) = \frac{1}{s} + \frac{1}{t} = 1.$$

We have equality if and only if the number sequences $(a_1^s, a_2^s, ..., a_n^s)$ and $(b_1^t, b_2^t, ..., b_n^t)$ are proportional. This follows from the conditions for equality in Theorem 1. $\square$

We will now show an additional useful result:

If we in Hölder's inequality allow one of the parameters p, q to be negative, then the inequality is reversed.

Suppose that $p < 0$, i.e. $s < 0$ with the notation above. We set $S = -\frac{s}{t}$ and $T = \frac{1}{t}$. Then we have $S > 0, T > 0$ and $\frac{1}{S} + \frac{1}{T} = 1$. Now we set $u_k = a_k^{-t}$ and $v_k = a_k^t b_k^t$, $(k = 1, 2, ..., n)$. Then Hölder's inequality yields $\sum_{k=1}^{n} u_k v_k \le \Big(\sum_{k=1}^{n} u_k^S\Big)^{\frac{1}{S}} \cdot \Big(\sum_{k=1}^{n} v_k^T\Big)^{\frac{1}{T}}$, which of course is equivalent to $\sum_{k=1}^{n} a_k^{-t} a_k^t b_k^t \le \Big(\sum_{k=1}^{n} a_k^{-t\cdot(-\frac{s}{t})}\Big)^{-\frac{t}{s}} \cdot \Big(\sum_{k=1}^{n} (a_k^t b_k^t)^{\frac{1}{t}}\Big)^{t}$. After some simplifying, rearranging and then taking the t:th root of both sides of the inequality, we get $\sum_{k=1}^{n} a_k b_k \ge \Big(\sum_{k=1}^{n} a_k^s\Big)^{\frac{1}{s}} \cdot \Big(\sum_{k=1}^{n} b_k^t\Big)^{\frac{1}{t}}$, which is the reversed inequality. □

6. Minkowski's inequality

We note that $\sum_{i=1}^{n} (a_i + b_i)^r = \sum_{i=1}^{n} (a_i + b_i)(a_i + b_i)^{r-1} = \sum_{i=1}^{n} a_i (a_i + b_i)^{r-1} + \sum_{i=1}^{n} b_i (a_i + b_i)^{r-1}$. For $r > 1$ we set s so that $\frac{1}{r} + \frac{1}{s} = 1$, i.e. $s = \frac{r}{r-1}$. Then, by Hölder's inequality, we have the following (we apply the inequality to both of the terms $\sum_{i=1}^{n} a_i (a_i + b_i)^{r-1}$ and $\sum_{i=1}^{n} b_i (a_i + b_i)^{r-1}$):

$\sum_{i=1}^{n} (a_i + b_i)^r \le \Big(\sum_{i=1}^{n} a_i^r\Big)^{\frac{1}{r}} \cdot \Big(\sum_{i=1}^{n} (a_i + b_i)^{(r-1)\cdot s}\Big)^{\frac{1}{s}} + \Big(\sum_{i=1}^{n} b_i^r\Big)^{\frac{1}{r}} \cdot \Big(\sum_{i=1}^{n} (a_i + b_i)^{(r-1)\cdot s}\Big)^{\frac{1}{s}} =$ $\Big(\sum_{i=1}^{n} a_i^r\Big)^{\frac{1}{r}} \cdot \Big(\sum_{i=1}^{n} (a_i + b_i)^r\Big)^{\frac{r-1}{r}} + \Big(\sum_{i=1}^{n} b_i^r\Big)^{\frac{1}{r}} \cdot \Big(\sum_{i=1}^{n} (a_i + b_i)^r\Big)^{\frac{r-1}{r}}$. If we then divide both sides of this inequality by $\Big(\sum_{i=1}^{n} (a_i + b_i)^r\Big)^{\frac{r-1}{r}}$, we get Minkowski's inequality.

It is obvious that Minkowski's inequality holds for $r = 1$; that condition yields equality. When $r > 1$ we have equality if and only if the number sequences a and b are proportional. This follows from the conditions for equality in Hölder's inequality.

If $r < 1$, $r \ne 0$, then s becomes negative, which leads to Minkowski's inequality getting reversed. This follows from the fact that Hölder's inequality becomes reversed for $s < 0$. □

7. Jensen's inequality

We will prove Jensen's inequality for rational positive numbers $\alpha_1, \alpha_2, ..., \alpha_n$ only. The general proof for nonnegative real numbers $\alpha_1, \alpha_2, ..., \alpha_n$ requires some serious analytical arguments.

We will divide the proof of into two cases.

Case 1: $\alpha_i = \frac{1}{n}$, $(i = 1, 2, ..., n)$. (We use the same approach as in the first proof of the AM-GM inequality.)

In this case we are supposed to prove the inequality

(1) $f\Big(\frac{1}{n}\sum_{i=1}^{n} x_i\Big) \le \frac{1}{n}\sum_{i=1}^{n} f(x_i)$. It is true for $n = 2$. This follows directly from the definition of convexity. Now, assume that (1) holds for $n = 2^k$, $(k = 1, 2, ...)$. Then (1) also holds for $m = 2^{k+1} = 2n$.

Proof: Let $x_1, x_2, ..., x_m \in I$. Then we have

$$f\Big(\frac{x_1 + x_2 + ... + x_m}{m}\Big) = f\Big(\frac{\frac{1}{n}\sum_{i=1}^{n} x_i + \frac{1}{n}\sum_{i=1}^{n} x_{n+i}}{2}\Big) \le \frac{f\Big(\frac{1}{n}\sum_{i=1}^{n} x_i\Big) + f\Big(\frac{1}{n}\sum_{i=1}^{n} x_{n+i}\Big)}{2} \le \frac{\sum_{i=1}^{n} f(x_i) + \sum_{i=1}^{n} f(x_{n+i})}{2n} = \frac{\sum_{i=1}^{m} f(x_i)}{m}.$$

In the expression above, the first inequality holds because (1) holds for $n = 2$. (We have that the two numbers $\frac{1}{n}\sum_{i=1}^{n} x_i$, $\frac{1}{n}\sum_{i=1}^{n} x_{n+i} \in I$, since they are the arithmetic mean of n numbers that all $\in I$.) The second inequality follows from our assumption. Thus, since (1) holds for $n = 2$, by induction it holds for $n = 2^k$, $(k = 1, 2, ...)$.

Assume that (1) holds for $n > 2$. Then (1) also holds for $n - 1$

Proof: Let $x_1, x_2, ..., x_{n-1} \in I$. For these numbers together with the number $x_n = \frac{1}{n-1}(x_1 + x_2 + ... + x_{n-1})$ (the arithmetic mean of $x_1, x_2, ..., x_{n-1}$), (1) holds according to our assumption. We get

(2) $$f\Big(\frac{x_1 + x_2 + ... + x_{n-1} + \frac{x_1+x_2+...+x_{n-1}}{n-1}}{n}\Big) \le \frac{f(x_1) + f(x_2) + ... + f(x_{n-1}) + f\Big(\frac{x_1+x_2+...+x_{n-1}}{n-1}\Big)}{n}.$$

After simplifying, the left-hand side of (2) becomes $f\Big(\frac{x_1 + x_2 + ... + x_{n-1}}{n-1}\Big)$. Thus, $f\Big(\frac{x_1 + x_2 + ... + x_{n-1}}{n-1}\Big) \le \frac{1}{n}\sum_{i=1}^{n-1} f(x_i) + \frac{1}{n} f\Big(\frac{x_1 + x_2 + ... + x_{n-1}}{n-1}\Big)$. Further simplifying yields the inequality $f\Big(\frac{x_1 + x_2 + ... + x_{n-1}}{n-1}\Big) \le \frac{1}{n-1}\sum_{i=1}^{n-1} f(x_i)$. Now, by induction, (1) holds for $n \ge 2$. Hence, (1) is proved in case 1.

Case 2: $\alpha_1, \alpha_2, ..., \alpha_n$ are rational positive numbers.

Since $\alpha_1, \alpha_2, ..., \alpha_n$ are rational positive numbers, there is a natural number m and nonnegative integers $p_1, p_2, ..., p_n$ such that $m = p_1 + p_2 + ... + p_n$ and $\alpha_i = \dfrac{p_i}{m}$, $(i = 1, 2, ..., n)$. (To realize this, just rewrite the numbers $\alpha_1, \alpha_2, ..., \alpha_n$ so that they all have the same denominator.)

Case 1 yields

(3) $f\Big(\dfrac{(x_1 + ... + x_1) + ... + (x_n + ... + x_n)}{m}\Big) \leq$
$\dfrac{(f(x_1) + ... + f(x_1)) + ... + (f(x_n) + ... + f(x_n))}{m}$. (The first parenthesis in the nominator on the left-hand side of the inequality contains p_1 terms, the second parenthesis p_2 terms, and so on.) Now, (3) can be expressed as $f\Big(\dfrac{1}{m}\sum_{i=1}^{n} p_i x_i\Big) \leq \dfrac{1}{m}\sum_{i=1}^{n} p_i f(x_i)$. Hence, (1) is proved in case 2. $\square$

8. Power Mean inequality

We start by pointing out the obvious: for $k = m$ we have equality, which means that the inequality holds in that case. Now we assume instead that $k < m$. Then we have $\dfrac{m}{k} > 1$ and it follows that the function $f(x) = x^{\frac{m}{k}}$ is strictly convex for $x \geq 0$. (The second derivative, $f''(x) = \dfrac{m}{k}\Big(\dfrac{m}{k} - 1\Big)x^{\frac{m}{k}-2} > 0$ for $x > 0$). Now, since the numbers $a_1, a_2, ..., a_n$ are nonnegative, clearly the numbers $a_1^k, a_2^k, ..., a_n^k$ are nonnegative too. With help from Jensen's inequality we then get the inequality $\dfrac{1}{n}(a_1^k)^{\frac{m}{k}} + \dfrac{1}{n}(a_2^k)^{\frac{m}{k}} + ... + \dfrac{1}{n}(a_n^k)^{\frac{m}{k}} \geq \Big(\dfrac{1}{n}a_1^k + \dfrac{1}{n}a_2^k + ... + \dfrac{1}{n}a_n^k\Big)^{\frac{m}{k}}$, which, after some simplifying, becomes $\dfrac{1}{n}(a_1^m + a_2^m + ... + a_n^m) \geq \Big(\dfrac{1}{n}(a_1^k + a_2^k + ... + a_n^k)\Big)^{\frac{m}{k}}$. Taking the m:th root of both sides of this inequality then yields the Power Mean inequality.

The condition for equality follows from the condition for equality in Jensen's inequality. $\square$

9. Schur's inequality

We will derive Schur's inequality from a stronger theorem.

Theorem 2: If a, b, c, u, v, w are nonnegative real numbers and we have

(1) $a^{\frac{1}{p}} + c^{\frac{1}{p}} \leq b^{\frac{1}{p}}$ and
(2) $u^{\frac{1}{p+1}} + w^{\frac{1}{p+1}} \geq v^{\frac{1}{p+1}}$ then, if $p > 0$, we have
(3) $ubc - vca + wab \geq 0$.

Proof: We start with two pairs of nonnegative numbers: $(a^{\frac{1}{p+1}}, c^{\frac{1}{p+1}})$ and

$((uc)^{\frac{1}{p+1}}, (wa)^{\frac{1}{p+1}})$. Since we have $p > 0$, we have $\dfrac{1}{p+1} > 0$, $\dfrac{p}{p+1} > 0$ and $\dfrac{p}{p+1} + \dfrac{1}{p+1} = 1$.

Then Hölder's inequality yields $a^{\frac{1}{p+1}}(uc)^{\frac{1}{p+1}} + c^{\frac{1}{p+1}}(wa)^{\frac{1}{p+1}} \le$

$\le (a^{\frac{1}{p+1}\cdot\frac{p+1}{p}} + c^{\frac{1}{p+1}\cdot\frac{p+1}{p}})^{\frac{p}{p+1}} \cdot ((uc)^{\frac{1}{p+1}\cdot(p+1)} + (wa)^{\frac{1}{p+1}\cdot(p+1)})^{\frac{1}{p+1}}$, i.e.

$(ac)^{\frac{1}{p+1}}u^{\frac{1}{p+1}} + (ac)^{\frac{1}{p+1}}w^{\frac{1}{p+1}} \le (a^{\frac{1}{p}} + c^{\frac{1}{p}})^{\frac{p}{p+1}} \cdot (uc + wa)^{\frac{1}{p+1}}$. Taking the $(p+1)$:th power of both sides of this inequality, we get $ac(u^{\frac{1}{p+1}} + w^{\frac{1}{p+1}})^{p+1} \le (a^{\frac{1}{p}} + c^{\frac{1}{p}})^p \cdot (uc + wa)$. Now we use the conditions (1) and (2) in order to get $ac(v^{\frac{1}{p+1}})^{p+1} \le (b^{\frac{1}{p}})^p(uc + wa)$, which is equivalent to
(3) $ubc - vca + wab \ge 0$. □

Now we can assume that $0 \le z \le y \le x$. Then, using Theorem 2 (setting $p = 1$, $a = y - z$, $b = x - z$, $c = x - y$, $u = x^r$, $v = y^r$, $w = z^r$), we get $x^r(x-z)(x-y) - y^r(x-y)(y-z) + z^r(y-z)(x-z) \ge 0$, which is equivalent to Schur's inequality.

It is quite easy to show that we have equality if and only if and only if $x = y = z$ or if two of x, y, z are equal and the third is 0. The only one of the three terms on the left-hand side of Schur's inequality that can be negative is $y^r(y-x)(y-z)$. It is negative when we have $y < x$ and $y > z$. But with those conditions we see that $x^r(x-z)(x-y) > |y^r(y-x)(y-z)|$. Thus, to have equality we must have $y = x$ or $y = z$. In either of these two cases two of the three terms on the left-hand side of the inequality $= 0$. It follows that in order to have equality we must have $x = y = z$ or two of x, y, z must be equal and the third must be 0. □

10. Maclaurin's inequality

We start by proving a theorem that we need in order to establish MacLaurin's inequality.

Theorem 3: For $n \ge 2$, let $a_1, a_2, ..., a_n$ be positive real numbers that are not all equal. Also let $p_0 = 1$ and $p_r = \sum_{1 \le i1 < i2 < ... < ir \le n} \dfrac{a_{i1}a_{i2}...a_{ir}}{\binom{n}{r}}, r = 1, 2, ..., n.$

Then, for all r, $1 \le r < n$, we have $p_{r-1}p_{r+1} < p_r^2$.

Proof (by induction): For $n = 2$ we have $p_0 = 1$, $p_1 = \dfrac{a_1 + a_2}{2}$ and $p_2 = a_1a_2$. Thus, $p_0p_2 = a_1a_2 < \left(\dfrac{a_1 + a_2}{2}\right)^2 = p_1^2$. Of course the inequality above follows from the AM-GM inequality (since we have $a_1 \ne a_2$, we have a strict inequality). Hence we have proved the theorem for $n = 2$.

Suppose now that the theorem holds for some $n = k - 1$, $k \ge 3$. We will show that it then holds for $n = k$.

According to our assumption, for the positive real numbers $a_1, a_2, ..., a_{k-1}$ that are not all equal we have $P_{r-1}P_{r+1} < P_r^2$, $1 \le r < k - 1$. (Here we write P instead of p to be able to separate these numbers that form the inequality for $n = k - 1$ from the numbers that form the inequality for $n = k$; the inequality that we are supposed to prove.) We want to show that for the positive real numbers $a_1, a_2, ..., a_k$ that are not all equal, the inequality $p_{r-1}p_{r+1} < p_r^2$, $1 \le r < k$, holds.

Now observe that $p_r = \frac{k-r}{k}P_r + \frac{r}{k}a_k P_{r-1}$, $r = 1, 2, ..., k$, where we set $P_k = 0$. (This can be a little tricky to realize, but it follows from the fact that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.) We get

(1) $k^2(p_{r-1}p_{r+1} - p_r^2) = k^2\Big(\Big(\frac{k-r+1}{k}P_{r-1} + \frac{r-1}{k}a_k P_{r-2}\Big)\Big(\frac{k-r-1}{k}P_{r+1} + \frac{r+1}{k}a_k P_r\Big) - \Big(\frac{k-r}{k}P_r + \frac{r}{k}a_k P_{r-1}\Big)^2\Big) = A + Ba_k + Ca_k^2$, where

$A = ((k-r)^2 - 1)P_{r-1}P_{r+1} - (k-r)^2 P_r^2$,

$B = (k-r+1)(r+1)P_{r-1}P_r + (k-r-1)(r-1)P_{r-2}P_{r+1} - 2(k-r)rP_rP_{r-1}$,

$C = (r^2-1)P_{r-2}P_r - r^2P_{r-1}^2$.

Since not all $a_1, a_2, ..., a_{k-1}$ are equal, we have $P_{r-1}P_{r+1} < P_r^2$, $P_{r-2}P_r < P_{r-1}^2$ and $P_{r-2}P_{r+1} < P_{r-1}P_r$. (The last inequality is valid because $P_{r-2}P_{r+1} = P_{r-2}P_r\frac{P_{r+1}}{P_r} < P_{r-1}^2\frac{P_{r+1}}{P_r} = \frac{P_{r-1}P_{r-1}P_{r+1}}{P_r} < \frac{P_{r-1}P_r^2}{P_r} = P_{r-1}P_r$.)

From these inequalities follow that $A < -P_r^2$, $B < 2P_{r-1}P_r$ and $C < -P_{r-1}^2$. Furthermore, from (1) we get $k^2(p_{r-1}p_{r+1} - p_r^2) < -P_r^2 + 2P_rP_{r-1}a_k - P_{r-1}^2a_k^2 = -(P_r - P_{r-1}a_k)^2 \le 0$. Thus, $k^2(p_{r-1}p_{r+1} - p_r^2) < 0$, which is equivalent to $p_{r-1}p_{r+1} < p_r^2$. Hence the theorem is proved for all $n \ge 2$.

One more thing remains, though; the case where $a_1 = a_2 = ... = a_{k-1} \neq a_k$. In that case we have $a_1 = \frac{P_r}{P_{r-1}}$, and from **(1)** we get $k^2(p_{r-1}p_{r+1} - p_r^2) = -a_1^2P_{r-1}^2 + 2a_1P_{r-1}^2a_k - P_{r-1}^2a_k^2 = -(a_1 - a_k)P_{r-1}^2 < 0$. Thus the theorem is proved in that case also. □

Proof of MacLaurin's inequality: Consider the inequalitites

$p_0p_2 < p_1^2$

$(p_1p_3)^2 < p_2^4$

$(p_2p_4)^3 < p_3^6$

.

.

.

$(p_{r-1}p_{r+1})^r < p_r^{2r}$

If we multiply together all the left-hand and right-hand sides respectively, we come up with the inequality $p_0p_1^2p_2^4p_3^6...p_{r-1}^{2r-2}p_r^{r-1}p_{r+1}^r < p_1^2p_2^4p_3^6...p_{r-1}^{2r-2}p_r^{2r}$, which yields $p_{r+1}^r < p_r^{r+1}$. The last inequality can also be written as $\sqrt[r]{p_r} > \sqrt[r+1]{p_{r+1}}$. Hence, we have $p_1 > \sqrt{p_2} > \sqrt[3]{p_3} > ... >$

$\sqrt[n-1]{p_{n-1}} > \sqrt[n]{p_n}$, which is MacLaurin's inequality.

In the case where $a_1 = a_2 = ... = a_n$, we have $p_1 = \sqrt{p_2} = \sqrt[3]{p_3} = ... = \sqrt[n-1]{p_{n-1}} = \sqrt[n]{p_n}$. This is easy to show. In that case we have $\sqrt[k-1]{p_{k-1}} = \sqrt[k-1]{\frac{1}{\binom{n}{k-1}}\binom{n}{k-1}a_1^{k-1}} = a_1 = \sqrt[k]{\frac{1}{\binom{n}{k}}\binom{n}{k}a_1} = \sqrt[k]{\frac{1}{\binom{n}{k}}\binom{n}{k}a_1^k} = \sqrt[k]{p_k}$. □

11. Muirhead's inequality

We start by proving the following lemma:

Lemma: Let a_1, a_2, b_1, b_2 be nonnegative real numbers such that $a_1 + a_2 = b_1 + b_2$ and max$(a_1, a_2)\geq$max(b_1, b_2). Let also x, y be nonnegative real numbers. Then we have

(0) $x^{a_1}y^{a_2} + x^{a_2}y^{a_1} \geq x^{b_1}y^{b_2} + x^{b_2}y^{b_1}$.

Proof: Because of symmetry we can without loss of generality assume that $a_1 \geq a_2$ and $b_1 \geq b_2$. If any of the numbers $x, y = 0$ then (0) obviously holds. Therefore we assume that $x, y \neq 0$. We have $x^{a_1}y^{a_2} + x^{a_2}y^{a_1} - x^{b_1}y^{b_2} - x^{b_2}y^{b_1} = x^{a_2}y^{a_2}(x^{a_1-a_2} + y^{a_1-a_2} - x^{b_1-a_2}y^{b_2-a_2} - x^{b_2-a_2}y^{b_1-a_2}) = x^{a_2}y^{a_2}(x^{b_1-a_2} - y^{b_1-a_2})(x^{b_2-a_2} - y^{b_2-a_2}) \geq 0$.

Now, why does $x^{a_2}y^{a_2}(x^{b_1-a_2} - y^{b_1-a_2})(x^{b_2-a_2} - y^{b_2-a_2}) \geq 0$ hold? Clearly, we have $x^{a_2}y^{a_2} \geq 0$. Then, from the conditions the inequalities $b_1 - a_2 \geq 0$ and $b_2 - a_2 \geq 0$ follow. If we then have $x \geq y$, we have $x^{b_1-a_2} - y^{b_1-a_2} \geq 0$ and $x^{b_2-a_2} - y^{b_2-a_2} \geq 0$, and the inequality follows. If we on the contrary have $x \leq y$, we have $x^{b_1-a_2} - y^{b_1-a_2} \leq 0$ and $x^{b_2-a_2} - y^{b_2-a_2} \leq 0$, and the inequality follows in that case too. □

We divide the proof of Muirhead's inequality into two cases.

Case 1: $b_1 \geq a_2$.

According to the conditions, we have $a_1 \geq a_1 + a_2 - b_1$ and $a_1 \geq b_1$. Thus, max$(a_1, a_2)\geq$ max$(a_1 + a_2 - b_1, b_1)$. We also note that $a_1 + a_2 = (a_1 + a_2 - b_1) + b_1$. We use the lemma and get $x^{a_1}y^{a_2} + x^{a_2}y^{a_1} \geq x^{a_1+a_2-b_1}y^{b_1} + x^{b_1}y^{a_1+a_2-b_1}$. Clearly also $z^{a_3}(x^{a_1}y^{a_2} + x^{a_2}y^{a_1}) \geq z^{a_3}(x^{a_1+a_2-b_1}y^{b_1} + x^{b_1}y^{a_1+a_2-b_1})$ holds. Then we realize that

(1) $\sum_{cyclic} z^{a_3}(x^{a_1}y^{a_2} + x^{a_2}y^{a_1}) \geq \sum_{cyclic} z^{a_3}(x^{a_1+a_2-b_1}y^{b_1} + x^{b_1}y^{a_1+a_2-b_1})$ holds too. (When we used the lemma, we could for example have used the numbers y and z instead of the numbers x and y and then multiplied both sides of the inequality with x^{a_3} instead of z^{a_3}.)

Now, we note that $\sum_{cyclic} z^{a_3}(x^{a_1}y^{a_2} + x^{a_2}y^{a_1}) = \sum_{sym} x^{a_1}y^{a_2}z^{a_3}$.

According to the conditions, we also have $a_1 + a_2 - b_1 \geq b_2 \geq b_3$, and max$(a_1 + a_2 - b_1, a_3)\geq$ max(b_2, b_3) follows. We note that $(a_1 + a_2 - b_1) + a_3 = b_2 + b_3$. Using the lemma yet another time yields $y^{a_1+a_2-b_1}z^{a_3} + y^{a_3}z^{a_1+a_2-b_1} \geq y^{b_2}z^{b_3} + y^{b_3}z^{b_2}$. Analogous to previous arguments one then realizes that

(2) $\quad \sum_{cyclic} x^{b_1}(y^{a_1+a_2-b_1}z^{a_3} + y^{a_3}z^{a_1+a_2-b_1}) \geq \sum_{cyclic} x^{b_1}(y^{b_2}z^{b_3} + y^{b_3}z^{b_2})$ also holds. Then we note that $\sum_{cyclic} x^{b_1}(y^{b_2}z^{b_3} + y^{b_3}z^{b_2}) = \sum_{sym} x^{b_1}y^{b_2}z^{b_3}$. At last we note that the right-hand side of (1) is equal to the left-hand side of (2). Then, from (1) and (2), Muirhead's inequality follows.

Case 2: $b_1 \leq a_2$.

According to the conditions, we have $3b_1 \geq b_1 + b_2 + b_3 = a_1 + a_2 + a_3 \geq b_1 + a_2 + a_3$. From this follows that $b_1 \geq a_2 + a_3 - b_1$. According to the conditions, we also have $a_2 \geq b_1$. Hence, max$(a_2, a_3)$$\geq$ max$(b_1, a_2 + a_3 - b_1)$. We note that $a_2 + a_3 = b_1 + (a_2 + a_3 - b_1)$. Now, we use the lemma to get $y^{a_2}z^{a_3} + y^{a_3}z^{a_2} \geq y^{b_1}z^{a_2+a_3-b_1} + y^{a_2+a_3-b_1}z^{b_1}$. Analogous with case 1, we see without problem that

(3) $\quad \sum_{cyclic} x^{a_1}(y^{a_2}z^{a_3} + y^{a_3}z^{a_2}) \geq \sum_{cyclic} x^{a_1}(y^{b_1}z^{a_2+a_3-b_1} + y^{a_2+a_3-b_1}z^{b_1})$ holds. We note that $\sum_{cyclic} x^{a_1}(y^{a_2}z^{a_3} + y^{a_3}z^{a_2}) = \sum_{sym} x^{a_1}y^{a_2}x^{a_3}$.

According to the conditions, we also have max$(a_1, a_2 + a_3 - b_1)$$\geq$ max(b_2, b_3). We note that $a_1 + (a_2 + a_3 - b_1) = b_2 + b_3$ and use the lemma one final time to get

$x^{a_1}z^{a_2+a_3-b_1} + x^{a_2+a_3-b_1}z^{a_1} \geq x^{b_2}z^{b_3} + x^{b_3}z^{b_2}$. Finally we note that

(4) $\quad \sum_{cyclic} y^{b_1}(x^{a_1}z^{a_2+a_3-b_1} + x^{a_2+a_3-b_1}z^{a_1}) \geq \sum_{cyclic} y^{b_1}(x^{b_2}z^{b_3} + x^{b_3}z^{b_2})$ holds.

We see that $\sum_{cyclic} y^{b_1}(x^{b_2}z^{b_3} + x^{b_3}z^{b_2}) = \sum_{sym} x^{b_1}y^{b_2}z^{b_3}$. Analogous with case 1 we see that the right-hand side of (3) is equal to the left-hand side of (4). Thus, from (3) and (4), Muirhead's inequality follows. □

SOLUTIONS TO THE PROBLEMS 1

1. Using the hint, we realize that $(x + y^3)(y + z^3)(z + x^3) \geq (x + 4y)(y + 4z)(z + 4x) = (x+y+y+y+y)(y+z+z+z+z)(z+x+x+x+x)$. Then, applying the AM-GM inequality to each of the parenthesis, we get $(x + y + y + y + y)(y + z + z + z + z)(z + x + x + x + x) \geq 5\sqrt[5]{xy^4} \cdot 5\sqrt[5]{yz^4} \cdot 5\sqrt[5]{zx^4} = 125xyz$.

2. Applying the AM-GM inequality to the denominator of each term on the left-hand side of the inequality, we get $\frac{x}{x^4 + y^2} + \frac{y}{y^4 + x^2} \leq \frac{x}{2\sqrt{x^4y^2}} + \frac{y}{2\sqrt{y^4x^2}} = \frac{1}{2xy} + \frac{1}{2xy} = \frac{1}{xy}$.

3. We apply the AM-GM inequality to the left-hand side of the inequality and get $\frac{1}{a}+\frac{4}{b}+\frac{4}{c}+\frac{16}{d} \geq 4\sqrt[4]{\frac{256}{abcd}} = \frac{16}{\sqrt[4]{abcd}}$. Then, using the AM-GM inequality again, this time on the denominator, we get $\frac{16}{\sqrt[4]{abcd}} \geq \frac{16}{\frac{a+b+c+d}{4}} = \frac{64}{a+b+c+d}$.

4. We have $\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right) = 1+\frac{c}{a}+\frac{b}{c}+\frac{b}{a}+\frac{a}{b}+\frac{c}{b}+\frac{a}{c}+1 = -1+\left(\frac{a}{a}+\frac{a}{b}+\frac{a}{c}\right)+\left(\frac{b}{b}+\frac{b}{a}+\frac{b}{c}\right)+\left(\frac{c}{c}+\frac{c}{a}+\frac{c}{b}\right)$. Then, applying the AM-GM inequality to each of the parenthesis, we get $-1+\left(\frac{a}{a}+\frac{a}{b}+\frac{a}{c}\right)+\left(\frac{b}{b}+\frac{b}{a}+\frac{b}{c}\right)+\left(\frac{c}{c}+\frac{c}{a}+\frac{c}{b}\right) \geq -1+3\frac{a+b+c}{\sqrt[3]{abc}}$. Now, all we have to show is that $-1+3\frac{a+b+c}{\sqrt[3]{abc}} \geq 2\left(1+\frac{a+b+c}{\sqrt[3]{abc}}\right)$, which is the same as showing that $\frac{a+b+c}{\sqrt[3]{abc}} \geq 3$. This inequality is obviously true: it is just a restatement of the AM-GM inequality for three terms.

5. We start, as suggested, by showing that $x^2+y^2+z^2 \geq xy+yz+zx$. We make the variable substitution $x^2=a$, $y^2=b$ and $z^2=c$. Then we have to show that $a+b+c \geq \sqrt{ab}+\sqrt{bc}+\sqrt{ca}$. Using the AM-GM inequality on each of the terms on the right-hand side of the inequality, we get $\sqrt{ab}+\sqrt{bc}+\sqrt{ca} \leq \frac{a+b}{2}+\frac{b+c}{2}+\frac{c+a}{2} = a+b+c$. (Those familiar with the Rearrangement inequality may think that the proof above is superfluous, since the proven inequality is a simple consequence of the Rearrangement inequality.)

Now we can show that $x^2+y^2+z^2+xy+yz+xz \geq 2(\sqrt{x}+\sqrt{y}+\sqrt{z})$. By using our hint inequality, we see that $x^2+y^2+z^2+xy+yz+xz \geq 2(xy+yz+xz) = 2\left(\frac{xy+xz}{2}+\frac{xy+yz}{2}+\frac{xz+yz}{2}\right)$. Then we apply the AM-GM inequality to each term inside the parenthesis to get $2\left(\frac{xy+xz}{2}+\frac{xy+yz}{2}+\frac{xz+yz}{2}\right) \geq 2(\sqrt{x\cdot xyz}+\sqrt{y\cdot xyz}+\sqrt{z\cdot xyz}) = 2(\sqrt{2x}+\sqrt{2y}+\sqrt{2z}) = 2\sqrt{2}(\sqrt{x}+\sqrt{y}+\sqrt{z}) \geq 2(\sqrt{x}+\sqrt{y}+\sqrt{z})$.

6. We start with the hint $1-y_i = \sum_{k\neq i} y_k$. Using the AM-GM inequality on the right-hand side of this inequality, we get $1-y_i \geq (n-1)\sqrt[n-1]{\prod_{k\neq i} y_k}$. Of course this is true for all i ($i = 1,2,...,n$), which means that we have n inequalities like the one above. We multiply together all the right-hand sides of these n equalities and treat similarly all the left-hand sides. Then we come up with the inequality $\prod_{i=1}^{n}(1-y_i) \geq (n-1)^n \prod_{i=1}^{n} y_i$. Dividing both sides by $\prod_{i=1}^{n} y_i$ yields the inequality $\prod_{i=1}^{n} \frac{1-y_i}{y_i} \geq (n-1)^n$. Now, $\frac{1-y_i}{y_i} = \frac{1-\frac{1998}{x_i+1998}}{\frac{1998}{x_i+1998}} = \frac{x_i+1998-1998}{1998} = \frac{x_i}{1998}$. Thus, our inequality becomes $\prod_{i=1}^{n} \frac{x_i}{1998} \geq (n-1)^n$, which after some easy rearranging becomes the inequality that we set out to prove.

7. For proof of the hint inequality, see problem 5. Using the AM-HM inequality on the left-hand side of the inequality, we get $\frac{1}{1+ab} + \frac{1}{1+bc} + \frac{1}{1+ca} \geq \frac{9}{3+ab+bc+ca}$. Then, applying the hint inequality, we get $\frac{9}{3+ab+bc+ca} \geq \frac{9}{3+a^2+b^2+c^2} = \frac{9}{6} = \frac{3}{2}$.

8. We start by showing the inequality $t + \frac{1}{t} \geq 2$ $(t > 0)$. We apply the AM-GM inequality to the left-hand side of this inequality to get $t + \frac{1}{t} \geq 2\sqrt{t \cdot \frac{1}{t}} = 2$.

Now to the main inequality. We note that all the denominators are positive. Then we subtract x_n from both sides of the inequality and express the left-hand side of the inequality in a clever way. We now have to prove the inequality

$$(x_0 - x_1) + \frac{1}{x_0 - x_1} + (x_1 - x_2) + \frac{1}{x_1 - x_2} + ... + (x_{n-1} - x_n) + \frac{1}{x_{n-1} - x_n} \geq 2n.$$

This becomes easy with the help of our hint inequality. The left-hand side of the above inequality consists of $2n$ terms. If we add the two first terms together, and then add the two following terms together, and so on, then, according to our hint inequality, the above inequality holds.

9. We start by multiplying both sides of the equality by $(4x^2 + y^2)$. Then, using the AM-GM inequality to each parenthesis on the right-hand side, we get

$$64x^2y^2 = (4x^2 + y^2)(x+1)(y+2)(2x+y) \geq 2\sqrt{4x^2 \cdot y^2} \cdot 2\sqrt{x} \cdot 2\sqrt{2y} \cdot 2\sqrt{2xy} = 64x^2y^2.$$

Hence the inequality above is an equality and the equality condition in AM-GM implies that $x = 1$ and $y = 2$.

10. We follow the hint and apply the AM-GM inequality three times; to the left-hand side of the equality on the factors $(1+2ax) = (1+ax+ax)$ and $(1+2by) = (1+by+by)$, and then to the whole right-hand side. We get the following: $\frac{27xy}{(1+2ax)(1+2by)} \leq \frac{27xy}{3\sqrt[3]{a^2x^2} \cdot 3\sqrt[3]{b^2y^2}} = 3\sqrt[3]{\frac{xy}{a^2b^2}}$ and $\frac{1}{ab} + \frac{x}{a} + \frac{y}{b} \geq 3\sqrt[3]{\frac{xy}{a^2b^2}}$. Hence $\frac{27xy}{(1+2ax)(1+2by)} \leq 3\sqrt[3]{\frac{xy}{a^2b^2}} \leq \frac{1}{ab} + \frac{x}{a} + \frac{y}{b}$. Now, to have equality, the condition for equality in the AM-GM inequality has to be met. This condition leads to

(1) $ax = by = 1$ and

(2) $\frac{1}{ab} = \frac{x}{a} = \frac{y}{b}$.

(1) yields $x = \frac{1}{a}$, $y = \frac{1}{b}$ while from (2), we get $x = \frac{1}{b}$, $y = \frac{1}{a}$. Hence, if we have $a \neq b$, then there is no pair of positive real numbers x, y such that $\frac{27xy}{(1+2ax)(1+2by)} = \frac{1}{ab} + \frac{x}{a} + \frac{y}{b}$. But if we have $a = b$, then we have equality for $x = y = \frac{1}{a}$.

11. We start by multiplying both sides of the inequality by $(a^n + b^n + c^n)$. Because of symmetry, we can, without loss of generality, assume that $a \leq b \leq c$. Then $a^n \geq b^n \geq c^n$ and the Chebyshev's inequality implies $\frac{1}{3}(a^n + b^n + c^n) \cdot \frac{1}{3}(a+b+c) \leq \frac{1}{3}(a^{n+1} + b^{n+1} + c^{n+1})$, which is the inequality in question.

12. We take logarithms on both sides and then use the well-known logarithm laws to rewrite the inequality. Since logarithm is a strictly increasing function, the inequality is not affected. We get $a \ln a + b \ln b + c \ln c \geq \frac{a+b+c}{3}(\ln a + \ln b + \ln c)$, which is the the inequality we want to prove.

Because of symmetry, we can once again without loss of generality assume that $a \leq b \leq c$. Then the inequality is an immediate consequence of the Chebyshev's inequality.

13. The left-hand side of the inequality can be rewritten as $a_1 \cdot \frac{1}{1} + a_2 \cdot \frac{1}{2^2} + ... + a_n \cdot \frac{1}{n^2}$. It is obvious that $\frac{1}{1} \geq \frac{1}{2^2} \geq ... \geq \frac{1}{n^2}$. Then, according to the Rearrangement inequality, $a_1 \cdot \frac{1}{1} + a_2 \cdot \frac{1}{2^2} + ... + a_n \cdot \frac{1}{n^2}$ is minimal when we have $a_1 < a_2 < ... < a_n$. Since all numbers a_i are distinct, then $a_i \geq i$. Thus we have $a_1 \cdot \frac{1}{1} + a_2 \cdot \frac{1}{2^2} + ... + a_n \cdot \frac{1}{n^2} \geq 1 \cdot \frac{1}{1} + 2 \cdot \frac{1}{2^2} + ... + n \cdot \frac{1}{n^2} = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$.

14. The left-hand side of the inequality can be rewritten as $x_1 \cdot \frac{1}{x_2} + x_2 \cdot \frac{1}{x_3} + ... + x_n \cdot \frac{1}{x_1}$. To minimize this expression with help from the Rearrangement inequality, the largest of the numbers $x_1, x_2, ..., x_n$ has to be multiplied by the smallest of the numbers $\frac{1}{x_1}, \frac{1}{x_2}, ..., \frac{1}{x_n}$, the second largest multiplied by the second smallest, and so on. Now, suppose that of all the numbers $x_1, x_2, ..., x_n$, the number x_k is the largest. Which one of the numbers $\frac{1}{x_1}, \frac{1}{x_2}, ..., \frac{1}{x_n}$ is then the smallest? $\frac{1}{x_k}$ of course! If the number x_l is the second largest then $\frac{1}{x_l}$ is the second smallest, and so on. Thus

$$a_1 \cdot \frac{1}{a_2} + a_2 \cdot \frac{1}{a_3} + ... + a_n \cdot \frac{1}{a_1} \geq a_1 \cdot \frac{1}{a_1} + a_2 \cdot \frac{1}{a_2} + ... + a_n \cdot \frac{1}{a_n} = n.$$

15. We have $\frac{a^8 + b^8 + c^8}{a^3 b^3 c^3} = a^5 \cdot \frac{1}{b^3 c^3} + b^5 \cdot \frac{1}{a^3 c^3} + c^5 \cdot \frac{1}{a^3 b^3}$. Because of the symmetry, we can without loss of generality assume that $a \geq b \geq c$. From that follows that $a^5 \geq b^5 \geq c^5$ and $\frac{1}{a^3 b^3} \leq \frac{1}{a^3 c^3} \leq \frac{1}{b^3 c^3}$. Then, using the Rearrangement inequality twice, we get

$a^5 \cdot \frac{1}{b^3c^3} + b^5 \cdot \frac{1}{a^3c^3} + c^5 \cdot \frac{1}{a^3b^3} \geq a^5 \cdot \frac{1}{a^3b^3} + b^5 \cdot \frac{1}{b^3c^3} + c^5 \cdot \frac{1}{a^3c^3} = a^2 \cdot \frac{1}{b^3} + b^2 \cdot \frac{1}{c^3} + c^2 \cdot \frac{1}{a^3} \geq$
$a^2 \cdot \frac{1}{a^3} + b^2 \cdot \frac{1}{b^3} + c^2 \cdot \frac{1}{c^3} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$

16. By the Rearrangement inequality we have $bz + cy \leq by + cz$. Hence $(by + cz)(bz + cy) \leq (by + cz) = b^2y^2 + 2bcyz + c^2z^2$. Moreover, since $2bcyz \leq b^2y^2 + c^2z^2$ then $(by + cz)(bz + cy) \leq 2(b^2y^2 + c^2z^2)$. From this follows that $\frac{a^2x^2}{(by + cz)(bz + cy)} \geq \frac{a^2x^2}{2(b^2y^2 + c^2z^2)}$.

A similar argument for the two other terms on the left hand side, together with the substitution $\alpha = a^2x^2$, $\beta = b^2y^2$, $\gamma = c^2z^2$ proves that the left-hand side is $\geq \frac{1}{2}(\frac{\alpha}{\beta + \gamma} + \frac{\beta}{\gamma + \alpha} + \frac{\gamma}{\alpha + \beta})$.

What we would need to show now is that the inequality $\frac{\alpha}{\beta + \gamma} + \frac{\beta}{\gamma + \alpha} + \frac{\gamma}{\alpha + \beta} \geq \frac{3}{2}$ holds. To that end multiply both sides of this inequality by (the positive number) $\alpha(\beta + \gamma) + \beta(\gamma + \alpha) + \gamma(\alpha + \beta)$. Then, using the Cauchy-Schwarz inequality on the left-hand side , we get

$\Big(\frac{\alpha}{\beta + \gamma} + \frac{\beta}{\gamma + \alpha} + \frac{\gamma}{\alpha + \beta}\Big) \cdot \big(\alpha(\beta + \gamma) + \beta(\gamma + \alpha) + \gamma(\alpha + \beta)\big) \geq$

$\Big(\sqrt{\frac{\alpha}{\beta + \gamma}} \cdot \sqrt{\alpha(\beta + \gamma)} + \sqrt{\frac{\beta}{\gamma + \alpha}} \cdot \sqrt{\beta(\gamma + \alpha)} + \sqrt{\frac{\gamma}{\alpha + \beta}} \cdot \sqrt{\gamma(\alpha + \beta)}\Big)^2 =$

$(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \alpha\gamma + \beta\gamma).$

On the right hand side of the inequality we have $\frac{3}{2}\big(\alpha(\beta + \gamma) + \beta(\gamma + \alpha) + \gamma(\alpha + \beta)\big) = 3(\alpha\beta + \beta\gamma + \alpha\gamma)$.

This means that what remains to show is that $\alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \alpha\gamma + \beta\gamma) \geq 3(\alpha\beta + \beta\gamma + \alpha\gamma)$, i.e. that $\alpha^2 + \beta^2 + \gamma^2 \geq \alpha\beta + \beta\gamma + \alpha\gamma$. For a simple proof, see problem 5.

17. Using the hint we note that $(a + b) + (b + c) + (c + d) + (d + a) = 2$ and we multiply both sides of the inequality by this factor. We get the inequality
$\big((a + b) + (b + c) + (c + d) + (d + a)\big)\Big(\frac{a^2}{a + b} + \frac{b^2}{b + c} + \frac{c^2}{c + d} + \frac{d^2}{d + a}\Big) \geq 1.$

Using the Cauchy-Schwarz inequality on the left-hand side yields $\big((a + b) + (b + c) + (c + d) + (d + a)\big)\Big(\frac{a^2}{a + b} + \frac{b^2}{b + c} + \frac{c^2}{c + d} + \frac{d^2}{d + a}\Big) \geq \Big(\sqrt{a + b} \cdot \sqrt{\frac{a^2}{a + b}} + \sqrt{b + c} \cdot \sqrt{\frac{b^2}{b + c}} + \sqrt{c + d} \cdot \sqrt{\frac{c^2}{c + d}} + \sqrt{d + a} \cdot \sqrt{\frac{d^2}{d + a}}\Big)^2 = (a + b + c + d)^2 = 1.$

To realize that we have equality if and only if $a = b = c = d = \frac{1}{4}$, we start by noting that when we have equality, the number sequences

$\big(\sqrt{a + b}, \sqrt{b + c}, \sqrt{c + d}, \sqrt{d + a}\big)$ and $\Big(\sqrt{\frac{a^2}{a + b}}, \sqrt{\frac{b^2}{b + c}}, \sqrt{\frac{c^2}{c + d}}, \sqrt{\frac{d^2}{d + a}}\Big)$ must be proportional. Thus when we have equality there is (and since all the numbers in these sequences are positive) a number $k > 0$ such that $k\sqrt{a+b} = \sqrt{\dfrac{a^2}{a+b}}$, $k\sqrt{b+c} = \sqrt{\dfrac{b^2}{b+c}}$, $k\sqrt{c+d} = \sqrt{\dfrac{c^2}{c+d}}$ and $k\sqrt{d+a} = \sqrt{\dfrac{d^2}{d+a}}$.

These four equations yield in turn $k = a = b = c = d$. Hence, when we have equality, we have $a = b = c = d = \dfrac{1}{4}$.

18. We start, as suggested, by multiplying both sides of the inequality by the factor $\big(a(b+2c) + b(c+2a) + c(a+2b)\big)$. We have then to prove that

$$\big(a(b+2c) + b(c+2a) + c(a+2b)\big) \cdot \Big(\frac{a}{b+2c} + \frac{b}{c+2a} + \frac{c}{a+2b}\Big) \geq$$
$$a(b+2c) + b(c+2a) + c(a+2b).$$

Let us apply the Cauchy-Schwarz inequality the the left-hand side of this inequality. We get

$$\big(a(b+2c) + b(c+2a) + c(a+2b)\big) \cdot \Big(\frac{a}{b+2c} + \frac{b}{c+2a} + \frac{c}{a+2b}\Big) \geq$$
$$\Big(\sqrt{a(b+2c)} \cdot \sqrt{\frac{a}{b+2c}} + \sqrt{(b(c+2a)} \cdot \sqrt{\frac{b}{c+2a}} + \sqrt{c(a+2b)} \cdot \sqrt{\frac{c}{a+2b}}\Big)^2 =$$
$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + ac + bc).$$

Working out the right-hand side of the inequality, we get $a(b+2c) + b(c+2a) + c(a+2b) = 3(ab + ac + bc)$. Once again we end up having to prove the inequality
$a^2 + b^2 + c^2 \geq ab + ac + bc$. See problem 5 for that proof.

19. We may begin by multiplying both sides of the inequality by $(ac+bd)$ to get the inequality $(ac+bd)\Big(\dfrac{a^3}{c} + \dfrac{b^3}{d}\Big) \geq ac + bd$. Then, applying the Cauchy-Schwarz inequality to the left-hand side yields $(ac+bd)\Big(\dfrac{a^3}{c} + \dfrac{b^3}{d}\Big) \geq \Big(\sqrt{ac} \cdot \sqrt{\dfrac{a^3}{c}} + \sqrt{bd} \cdot \sqrt{\dfrac{b^3}{d}}\Big)^2 = (a^2+b^2)^2$.

We now have to show that the inequality $(a^2+b^2)^2 \geq ac + bd$ holds. We square this inequality. Then we get $(a^2+b^2)^4 = (a^2+b^2)(a^2+b^2)^3 = (a^2+b^2)(c^2+d^2) \geq (ac+bd)^2$. The inequality $(a^2+b^2)(c^2+d^2) \geq (ac+bd)^2$ holds, of course, according to the Cauchy-Schwarz inequality.

20. For this problem we will give two solutions.

Solution 1 (using Hölder's inequality): We make the suggested variable substitution. The inequality to be proven becomes $(1+a_1^n)(1+a_2^n)...(1+a_n^n) \geq (1 + a_1a_2...a_n)^n$.

If we now take the n:th root of both sides and then write the inequality as $(1^n + a_1^n)^{\frac{1}{n}}(1^n + a_2^n)^{\frac{1}{n}}...(1^n + a_n^n)^{\frac{1}{n}} \geq (1 + a_1a_2...a_n)$. It is obvious that this inequality holds (Hölder's inequality for n number sequences).

Solution 2 (using the AM-GM inequality): Using the binomial theorem, we develop both sides of the inequality. We get the inequality

$1 + (x_1 + x_2 + ... + x_n) + (x_1x_2 + x_1x_3 + ... + x_{n-1}x_n) + ... + (x_1x_2...x_n) \geq$
$1 + \binom{n}{1}(x_1x_2...x_n)^{\frac{1}{n}} + \binom{n}{2}(x_1x_2...x_n)^{\frac{2}{n}} + ... + \binom{n}{n}(x_1x_2...x_n).$

Let us subtract 1 from both sides of the inequality. Then we have n parenthesis on the left-hand side of the inequality, and n terms on the right-hand side. If we are able to show that the k:th parenthesis on the left-hand side ($k = 1, 2, ..., n$) is $\geq$ the k:th term on the right-hand side, we are done. Now, the k:th parenthesis on the left-hand side consists of $\binom{n}{k}$ terms. The factor x_i, ($k = 1, 2, ..., n$) exists in $\binom{n-1}{k-1}$ of these terms. We use the AM-GM inequality on these $\binom{n}{k}$ terms to get that the parenthesis is $\geq \binom{n}{k}\left(x_1^{\binom{n-1}{k-1}}x_2^{\binom{n-1}{k-1}}...x_n^{\binom{n-1}{k-1}}\right)^{1/\binom{n}{k}}$.

We note that $\binom{n-1}{k-1} = \frac{(n-1)!}{(k-1)!(n-k)!}$, and that $1/\binom{n}{k} = \frac{k!(n-k)!}{n!} =$ $\frac{k}{n} \cdot \frac{(k-1)!(n-k)!}{(n-1)!}$. From this it follows that $\binom{n}{k}\left(x_1^{\binom{n-1}{k-1}}x_2^{\binom{n-1}{k-1}}...x_n^{\binom{n-1}{k-1}}\right)^{1/\binom{n}{k}} =$ $\binom{n}{k}(x_1x_2...x_n)^{\frac{k}{n}}$. Since $\binom{n}{k}(x_1x_2...x_n)^{\frac{k}{n}}$ is the k:th term on the right-hand side of the inequality, we are done.

21. Using Hölder's inequality (with $p = \frac{2}{3}$ and $q = \frac{1}{3}$) on the right-hand side of the inequality, we get $a^2b + b^2c + c^2a \leq \left((a^2)^{\frac{3}{2}} + (b^2)^{\frac{3}{2}} + (c^2)^{\frac{3}{2}}\right)^{\frac{2}{3}}(b^3 + c^3 + a^3)^{\frac{1}{3}} = a^3 + b^3 + c^3$.

22. Applying Hölder's inequality for three number sequences (these sequences being (a^p, b^p, c^p), and (a^q, b^q, c^q), (a^r, b^r, c^r)) to the right-hand side of the inequality yields $a^pb^qc^r + c^pa^qb^r + b^pc^qa^r \leq$ $\left((a^p)^{\frac{1}{p}} + (b^p)^{\frac{1}{p}} + (c^p)^{\frac{1}{p}}\right)^p \cdot \left((b^q)^{\frac{1}{q}} + (a^q)^{\frac{1}{q}} + (c^q)^{\frac{1}{q}}\right)^q \cdot \left((c^r)^{\frac{1}{r}} + (b^r)^{\frac{1}{r}} + (a^r)^{\frac{1}{r}}\right)^r =$ $(a + b + c)^{(p+q+r)} = a + b + c.$

23. We start by taking the third root of both sides of the inequality. Then we express x_k and $\frac{1}{x_k}$ as $(x_k^{\frac{1}{3}})^3$ and $\left(\left(\frac{1}{x_k}\right)^{\frac{2}{3}}\right)^{\frac{3}{2}}$ respectively. We come up with the inequality $2^{\frac{1}{3}}\left(\sum_{k=1}^{n}(x_k^{\frac{1}{3}})^3\right)^{\frac{1}{3}} \cdot \left(\sum_{k=1}^{n}\left(\left(\frac{1}{x_k}\right)^{\frac{2}{3}}\right)^{\frac{3}{2}}\right)^{\frac{2}{3}} \geq n$. Using first Hölder's inequality on the left-hand side of this inequality and then using the condition $x_k \in [1, 2]$, $k = 1, 2, ..., n$, we get $2^{\frac{1}{3}}\left(\sum_{k=1}^{n}(x_k^{\frac{1}{3}})^3\right)^{\frac{1}{3}} \cdot \left(\sum_{k=1}^{n}\left(\left(\frac{1}{x_k}\right)^{\frac{2}{3}}\right)^{\frac{3}{2}}\right)^{\frac{2}{3}} \geq$
$\geq 2^{\frac{1}{3}}\left(\sum_{k=1}^{n}\left(x_k^{\frac{1}{3}} \cdot \left(\frac{1}{x_k}\right)^{\frac{2}{3}}\right)\right) = 2^{\frac{1}{3}}\left(\sum_{k=1}^{n}\frac{1}{x_k^{\frac{1}{3}}}\right) \geq 2^{\frac{1}{3}}\left(\sum_{k=1}^{n}\frac{1}{2^{\frac{1}{3}}}\right) = n.$

24. We set $a_1 = \dfrac{x\sqrt{a}}{\sqrt{a+b+c}}$, $a_2 = \dfrac{y\sqrt{b}}{\sqrt{a+b+c}}$, $a_3 = \dfrac{z\sqrt{c}}{\sqrt{a+b+c}}$,
$b_1 = \dfrac{y\sqrt{a}}{\sqrt{a+b+c}}$, $b_2 = \dfrac{z\sqrt{b}}{\sqrt{a+b+c}}$, $b_3 = \dfrac{x\sqrt{c}}{\sqrt{a+b+c}}$ and
$c_1 = \dfrac{z\sqrt{a}}{\sqrt{a+b+c}}$, $c_2 = \dfrac{x\sqrt{b}}{\sqrt{a+b+c}}$, $c_3 = \dfrac{y\sqrt{c}}{\sqrt{a+b+c}}$,
and use Minkowski's inequality for three number sequences (a_1, a_2, a_3), (b_1, b_2, b_3), $(c_{,1}\,,_{,2}\,, c_3)$ and $r = 2$. We get

$$\sqrt{\frac{ax^2+by^2+cz^2}{a+b+c}} + \sqrt{\frac{ay^2+bz^2+cx^2}{a+b+c}} + \sqrt{\frac{az^2+bx^2+cy^2}{a+b+c}} = \sqrt{a_1^2+a_2^2+a_3^2}+$$
$$\sqrt{b_1^2+b_2^2+b_3^2} + \sqrt{c_1^2+c_2^2+c_3^2} \geq \sqrt{(a_1+b_1+c_1)^2+(a_2+b_2+c_2)^2+(a_3+b_3+c_3)^2} =$$
$$\sqrt{\Big(\frac{x\sqrt{a}+y\sqrt{a}+z\sqrt{a}}{\sqrt{a+b+c}}\Big)^2 + \Big(\frac{y\sqrt{b}+z\sqrt{b}+x\sqrt{b}}{\sqrt{a+b+c}}\Big)^2 + \Big(\frac{z\sqrt{c}+x\sqrt{c}+y\sqrt{c}}{\sqrt{a+b+c}}\Big)^2},$$

which after some simplification becomes $x+y+z$.

25. We rearrange the inequality and get the equivalent expression to prove

$$\Big(\sqrt{\frac{1}{a}+\frac{1}{b}+\frac{4}{c}+\frac{16}{d}} \cdot \sqrt{a+b+c+d}\Big)^2 \geq 64.$$

Now we can use Hölder's inequality on the left-hand side to get

$$\Big(\sqrt{\frac{1}{a}+\frac{1}{b}+\frac{4}{c}+\frac{16}{d}}\cdot\sqrt{a+b+c+d}\Big)^2 \geq \Big(\frac{1}{\sqrt{a}}\cdot\sqrt{a}+\frac{1}{\sqrt{b}}\cdot\sqrt{b}+\frac{2}{\sqrt{c}}\cdot\sqrt{c}+\frac{4}{\sqrt{d}}\cdot\sqrt{d}\Big)^2 = 8^2 = 64.$$

26. First we show that the left inequality holds. We note that $f(x) = \dfrac{1}{x}$ is strictly convex for $x > 0$ (we have $f''(x) = \dfrac{2}{x^3} > 0$ for $x > 0$). Then we divide both sides of the inequality by 6 to come up with the inequality $\dfrac{1,5}{a+b+c} \leq \dfrac{1}{3}\cdot\dfrac{1}{a+b} + \dfrac{1}{3}\cdot\dfrac{1}{b+c} + \dfrac{1}{3}\cdot\dfrac{1}{c+a}$. Using Jensen's inequality (with the function mentioned above) on the right-hand side of the inequality then yields

$$\frac{1}{3}\cdot\frac{1}{a+b}+\frac{1}{3}\cdot\frac{1}{b+c}+\frac{1}{3}\cdot\frac{1}{c+a} \geq \frac{1}{\frac{1}{3}(a+b)+\frac{1}{3}(b+c)+\frac{1}{3}(c+a)} = \frac{3}{2a+2b+2c} = \frac{1,5}{a+b+c}.$$

To show that the right-hand inequality holds we start by expressing the right-hand side of this inequality in a suitable way. Then, once again, we use Jensen's inequality; this time on each of the parenthesis (the function being the same as before). We get $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c} = \Big(\dfrac{1}{2}\cdot\dfrac{1}{a}+\dfrac{1}{2}\cdot\dfrac{1}{b}\Big)+$

$$\Big(\frac{1}{2}\cdot\frac{1}{b}+\frac{1}{2}\cdot\frac{1}{c}\Big)+\Big(\frac{1}{2}\cdot\frac{1}{c}+\frac{1}{2}\cdot\frac{1}{a}\Big) \geq \frac{1}{\frac{1}{2}a+\frac{1}{2}b}+\frac{1}{\frac{1}{2}b+\frac{1}{2}c}+\frac{1}{\frac{1}{2}c+\frac{1}{2}a} = 2\Big(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\Big).$$

27. We begin by taking a look at the left-hand side of the inequality. We rearrange this expres-

sion, getting $\frac{x}{x+1}+\frac{y}{y+1}+\frac{z}{z+1} = \frac{x+1-1}{x+1}+\frac{y+1-1}{y+1}+\frac{z+1-1}{z+1} = 3-\frac{1}{x+1}-\frac{1}{y+1}-\frac{1}{z+1}$. Our inequality to be proven is thus the following: $3-\frac{1}{x+1}-\frac{1}{y+1}-\frac{1}{z+1} \leq \frac{3}{4}$. After some easy rearranging we come up with the equivalent inequality
$\frac{3}{4} \leq \frac{1}{3}\Big(\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}\Big)$. We note once more that $f(x) = \frac{1}{x}$ is strictly convex for $x > 0$. Then we apply Jensen's inequality to the right-hand side of the last inequality and get $\frac{1}{3}\Big(\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}\Big) \geq \frac{1}{\frac{1}{3}(x+1)+\frac{1}{3}(y+1)+\frac{1}{3}(z+1)} = \frac{3}{3+x+y+z} = \frac{3}{4}$.

28. We set $f(x) = \frac{1}{4x-x^3}$. If one takes a look on $f''(x)$, one realizes that we have $f''(x) = \frac{12(x^2-1)^2+20}{x^3(4-x^2)^3} > 0$ for $0 < x < 2$, which means that $f(x)$ is strictly convex in this interval. We can then use Jensen's inequality. We get the following: $\sum_{k=1}^{n} \frac{1}{4a_k - a_k^3} = n \cdot \sum_{k=1}^{n} \frac{1}{n} \cdot \frac{1}{4a_k - a_k^3} \geq \frac{n}{4(\frac{1}{n}\cdot\sum_{k=1}^{n} a_k) - (\frac{1}{n}\cdot\sum_{k=1}^{n} a_k)^3} = \frac{n}{4(\frac{1}{n}\cdot n)-(\frac{1}{n}\cdot n)^3} = \frac{n}{3}$. Thus, the minimum value of the expression is $\frac{n}{3}$ and is attained when $a_1 = a_2 = ... = a_n = 1$.

29. If one draws an arbitrary triangle with an inscribed circle of radius 1 it is not difficult to realize that the hint holds.

Now, consider the function $f(x) = \frac{1}{\tan x}$. It is strictly convex for $0 < x < \frac{\pi}{2}$, which one realizes by studying $f''(x) = \frac{\sin 2x}{\sin^4 x} > 0$ for $0 < x < \frac{\pi}{2}$.

Using Jensen's inequality then yields $\frac{1}{\tan\alpha}+\frac{1}{\tan\beta}+\frac{1}{\tan\gamma} = 3\Big(\frac{1}{3}\cdot\frac{1}{\tan\alpha}+\frac{1}{3}\cdot\frac{1}{\tan\beta}+\frac{1}{3}\cdot\frac{1}{\tan\gamma}\Big) \geq \frac{3}{\tan(\frac{1}{3}(\alpha+\beta+\gamma))} = \frac{3}{\tan\frac{\pi}{6}} = \frac{1}{\tan\frac{\pi}{6}}+\frac{1}{\tan\frac{\pi}{6}}+\frac{1}{\tan\frac{\pi}{6}}$,
which is half the circumference of the equilateral triangle.

30. We make the suggested variable substitution to get the inequality
$\sqrt{\frac{e^a}{e^a+8}}+\sqrt{\frac{e^b}{e^b+8}}+\sqrt{\frac{e^c}{e^c+8}} \geq 1$. The conditions $0 < x, y, z < 4$ and $xyz = 1$ become the conditions $0 < e^a, e^b, e^c < 4$ and $a+b+c = 0$ respectively. Then we study the function $f(s) = \Big(\frac{e^s}{e^s+8}\Big)^{\frac{1}{2}}$. After some work we get $f'(s) = \frac{4e^{\frac{s}{2}}}{(e^s+8)^{\frac{3}{2}}}$ and $f''(s) = \frac{2e^{\frac{s}{2}}(8-2e^s)}{(e^s+8)^{\frac{5}{2}}}$. We see that $f''(s) > 0$ for $e^s < 4$. Hence $f(s)$ is strictly convex for $e^s < 4$. Now we multiply both sides of our inequality by $\frac{1}{3}$, and we get $\frac{1}{3}\sqrt{\frac{e^a}{e^a+8}}+\frac{1}{3}\sqrt{\frac{e^b}{e^b+8}}+\frac{1}{3}\sqrt{\frac{e^c}{e^c+8}} \geq \frac{1}{3}$.

Applying now Jensen's inequality to the left-hand side yields

$\frac{1}{3}\sqrt{\frac{e^a}{e^a+8}}+\frac{1}{3}\sqrt{\frac{e^b}{e^b+8}}+\frac{1}{3}\sqrt{\frac{e^c}{e^c+8}}\geq\sqrt{\frac{e^{\frac{1}{3}(a+b+c)}}{e^{\frac{1}{3}(a+b+c)}+8}}=\sqrt{\frac{1}{1+8}}=\frac{1}{3}$, and we are done.

31. We use the hint and find, according to the Power Mean inequality, that $\left(\frac{x^2+y^2}{2}\right)^{\frac{1}{2}}\leq\left(\frac{x^3+y^3}{2}\right)^{\frac{1}{3}}$, which is equivalent to the inequality $x^2+y^2\leq(x^3+y^3)^{\frac{2}{3}}\cdot 2^{\frac{1}{3}}$. Applying now the given condition $x^3+y^3>2$ to the right-hand side of this inequality yields $(x^3+y^3)^{\frac{2}{3}}\cdot 2^{\frac{1}{3}}<(x^3+y^3)^{\frac{2}{3}}\cdot(x^3+y^3)^{\frac{1}{3}}=x^3+y^3$. Thus we have

(1) $\quad x^2+y^2<x^3+y^3$.

We also have $(1-y)^2\geq 0$. Developing the left-hand side of this inequality and then multiplying both sides by y^2 yield

(2) $\quad y^2-2y^3+y^4\geq 0$.

Now, if we add the left-hand and the right-hand sides of (1) and (2) respectively, we come up with the desired inequality.

32. We start by showing that the inequality $\sum_{i=1}^{4}a_i^3\geq\sum_{i=1}^{4}a_i$. holds. To this end we divide both sides of the inequality by 4. Then we apply the Power Mean inequality to the right-hand side, getting $\frac{1}{4}(a_1+a_2+a_3+a_4)\leq\left(\frac{1}{4}({a_1}^3+{a_2}^3+{a_3}^3+{a_4}^3)\right)^{\frac{1}{3}}$. What we now have to show is that $\frac{1}{4}({a_1}^3+{a_2}^3+{a_3}^3+{a_4}^3)\geq\left(\frac{1}{4}({a_1}^3+{a_2}^3+{a_3}^3+{a_4}^3)\right)^{\frac{1}{3}}$. It is obvious that this inequality holds if the inequality
$\frac{1}{4}\left({a_1}^3+{a_2}^3+{a_3}^3+{a_4}^3\right)\geq 1$ holds. And it does: using the AM-GM inequality on the left-hand side yields $\frac{1}{4}({a_1}^3+{a_2}^3+{a_3}^3+{a_4}^3)\geq\sqrt[4]{{a_1}^3\cdot{a_2}^3\cdot{a_3}^3\cdot{a_4}^3}=1$.

The second inequality that we have to show is $\sum_{i=1}^{4}a_i^3\geq\sum_{i=1}^{4}\frac{1}{a_i}$. To show that, we set $A={a_1}^3+{a_2}^3+{a_3}^3+{a_4}^3$ and $A_i=A-{a_i}^3$, $(i=1,2,3,4)$. We see then that $\sum_{i=1}^{4}A_i=4A-\sum_{i=1}^{4}a_i^3=3A$. Thus, $\frac{1}{3}\sum_{i=1}^{4}A_i=A$. Now we note that, by the AM-GM inequality, we have $\frac{1}{3}A_1=\frac{1}{3}({a_2}^3+{a_3}^3+{a_4}^3)\geq\sqrt[3]{{a_2}^3{a_3}^3{a_4}^3}=a_2a_3a_4=\frac{1}{a_1}$. Using the very same reasoning, we realize that the inequalities $\frac{1}{3}A_2\geq\frac{1}{a_2}$, $\frac{1}{3}A_3\geq\frac{1}{a_3}$ and $\frac{1}{3}A_4\geq\frac{1}{a_4}$ also hold. Adding the left-hand and the right-hand sides respectively of these four inequalities then yields the desired inequality.

33. We begin by showing that $a^3+b^3+c^3\geq\frac{1}{3}(a+b+c)(a^2+b^2+c^2)$. We multiply both

sides of this inequality by $\frac{1}{3}$ and apply the Power Mean inequality to the right-hand side of the inequality then yields $\frac{1}{3}(a+b+c) \cdot \frac{1}{3}(a^2+b^2+c^2) \leq$
$\leq \left(\frac{1}{3}(a^3+b^3+c^3)\right)^{\frac{1}{3}} \cdot \left(\frac{1}{3}((a^2)^{\frac{3}{2}}+(b^2)^{\frac{3}{2}}+(c^2)^{\frac{3}{2}})\right)^{\frac{2}{3}} = \frac{1}{3}(a^3+b^3+c^3).$

Now let's turn to the inequality that we are supposed to show. Using the recently proved inequality on the left-hand side the inequality to be shown, we get

$$\frac{a^3+b^3+c^3}{a+b+c}+\frac{a^3+b^3+d^3}{a+b+d}+\frac{a^3+c^3+d^3}{a+c+d}+\frac{b^3+c^3+d^3}{b+c+d} \geq \frac{\frac{1}{3}(a+b+c)(a^2+b^2+c^2)}{a+b+c}+$$
$$\frac{\frac{1}{3}(a+b+d)(a^2+b^2+d^2)}{a+b+d}+\frac{\frac{1}{3}(a+c+d)(a^2+c^2+d^2)}{a+c+d}+\frac{\frac{1}{3}(b+c+d)(b^2+c^2+d^2)}{b+c+d}=$$
$$\frac{1}{3}(a^2+b^2+c^2)+\frac{1}{3}(a^2+b^2+d^2)+\frac{1}{3}(a^2+c^2+d^2)+\frac{1}{3}(b^2+c^2+d^2)=a^2+b^2+c^2+d^2.$$

34. We follow the hint and homogenize the inequality. On the left-hand side of the inequality we multiply the term $2\sqrt{3xyz}$ by $1 = \sqrt{x+y+z}$. Then we multiply the right-hand side by $1 = (x+y+z)^2$. We develop the right-hand side and simplify to come up with the inequality $\sqrt{3xyz} \cdot \sqrt{x+y+z} \leq xy+xz+yz$. We continue by squaring both sides and then getting rid of terms that cancel each other. We get the inequality $x^2yz+xy^2z+xyz^2 \leq x^2y^2+x^2z^2+y^2z^2$. To see that this inequality holds, we make the variable substitution $xy=a$, $xz=b$, $yz=c$. This leads to the, probably by now, familiar inequality $ab+ac+bc \leq a^2+b^2+c^2$, which holds. For proof of that, see problem 5.

35. In order to use Schur's inequality we need to make the given inequality homogeneous. This is done by multiplying the left-hand side of the inequality by $1=(p+q+r)$ and the term 2 on the right-hand side by $1=(p+q+r)^3$. Multiplying out and canceling some terms we end up with $p^2q+pq^2+p^2r+pr^2+q^2r+qr^2 \leq 2(p^3+q^3+r^3)$.

Rewriting the right-hand side of this expression and then using the AM-GM inequality on the parenthesis yields $2(p^3+q^3+r^3) = p^3+q^3+r^3+(p^3+q^3+r^3) \geq p^3+q^3+r^3+3\sqrt[3]{p^3q^3r^3} = p^3+q^3+r^3+3pqr$. What is now left for us to show is that the inequality $p^2q+pq^2+p^2r+pr^2+q^2r+qr^2 \leq p^3+q^3+r^3+3pqr$ holds. Not much to show though, since this is Schur's inequality for $r=1$.

36. We use the suggested notation and get $(1+a_1)(1+a_2)\cdot ... \cdot(1+a_n) = 1+\binom{n}{1}S_1+\binom{n}{2}S_2+...+\binom{n}{n}S_n$.

Now we will show that $1+\binom{n}{1}S_1+\binom{n}{2}S_2+...+\binom{n}{n}S_n \geq (1+S_n^{\frac{1}{n}})^n = .1+\binom{n}{1}S_n^{\frac{1}{n}}+\binom{n}{2}S_n^{\frac{2}{n}}+...+\binom{n}{n-1}S_n^{\frac{n-1}{n}}+\binom{n}{n}S_n$. In fact we find that for all $k=0,1,2,...,n$ we have $\binom{n}{k}S_k \geq \binom{n}{k}S_n^{\frac{k}{n}}$. This is true, since, by MacLaurin's inequality, $S_k^{\frac{1}{k}} \geq S_n^{\frac{1}{n}}$.

Now, what we have shown is that $2^n = (1+a_1)(1+a_2)\cdot\ldots\cdot(1+a_n) \geq (1+(a_1\cdot a_2\cdot\ldots\cdot a_n)^{\frac{1}{n}})^n$. Taking the n:th root of both sides, subtracting 1 and last taking the n:th power yields the desired inequality.

A maybe easier solution uses Hölder's inequality, as in problem 20.

37. We make the suggested substitutions $a = x^3$, $b = y^3$, $c = z^3$, multiply both sides by $x^3y^3z^3$ to get rid of the denominators, multiply out the parenthesis and we end up with the inequality $\sum\limits_{sym} x^6y^3z^0 \geq \sum\limits_{sym} x^5y^2z^2$. This is clearly true by Muirhead's inequality for $a_1 = 6$, $a_2 = 3$, $a_3 = 0$, $b_1 = 5$, $b_2 = 2$, $b_3 = 2$.

38. In order to get rid of the denominators we start by multiplying both sides by $4(x+y)(x+z)(y+z)(x+y+z)$. Then we multiply the parenthesis and cancel terms that occur on both sides. Eventually we get the inequality $6xyz \leq x^2y + x^2z + y^2x + y^2z + z^2x + z^2y$, which also can be expressed as $\sum\limits_{sym} xyz \leq \sum\limits_{sym} x^2yz^0$. It is easy to see that this inequality holds since it is Muirhead's inequality for $a_1 = 2$, $a_2 = 1$, $a_3 = 0$, $b_1 = b_2 = b_3 = 1$.

39. By making the trigonometric substitution $x = \cos\alpha$, $y = \cos\beta$, for $0 \leq \alpha, \beta \leq \pi$, and simplifying the expression using well-known trigonometric identities, we get

$f(\cos\alpha, \cos\beta) = \cos\alpha\cos\beta + \cos\alpha\sqrt{1-\cos^2\beta} + \cos\beta\sqrt{1-\cos^2\alpha} -$

$\sqrt{(1-\cos^2\alpha)(1-\cos^2\beta)} = \cos\alpha\cos\beta + \cos\alpha\sqrt{\sin^2\beta} + \cos\beta\sqrt{\sin^2\alpha} - \sqrt{\sin^2\alpha\sin^2\beta} =$

$\cos\alpha\cos\beta + \cos\alpha\sin\beta + \cos\beta\sin\alpha - \sin\alpha\sin\beta = \cos(\alpha+\beta) + \sin(\alpha+\beta) =$

$\sqrt{2}\Big(\frac{1}{\sqrt{2}}\cos(\alpha+\beta) + \frac{1}{\sqrt{2}}\sin(\alpha+\beta)\Big) = \sqrt{2}\Big(\sin\frac{\pi}{4}\cos(\alpha+\beta) + \cos\frac{\pi}{4}\sin(\alpha+\beta)\Big) =$

$\sqrt{2}\sin\Big(\frac{\pi}{4} + \alpha + \beta\Big) \leq \sqrt{2}$.

The equality is attained for all α, β. such that $\alpha+\beta = \frac{\pi}{4}$. Thus, for real numbers $-1 \leq x, y \leq 1$, the function $f(x, y)$ has the maximum value $\sqrt{2}$ and the maximum is attained for all $x = \arccos\alpha$, $y = \arccos\beta$ such that $\alpha + \beta = \frac{\pi}{4}$.

40. We make the suggested variable substitution $x = \tan\frac{\alpha}{2}$, $y = \tan\frac{\beta}{2}$ and $z = \tan\frac{\gamma}{2}$, where $0 < \alpha, \beta, \gamma < \pi$. Using some well-known trigonometric identities we eventually reduce the inequality to

(1) $\quad \sin 2\alpha + \sin 2\beta + \sin 2\gamma \leq \sin\alpha + \sin\beta + \sin\gamma$.

Now, the condition $xy + yz + zx = 1$ becomes $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} + \tan\frac{\beta}{2}\tan\frac{\gamma}{2} + \tan\frac{\gamma}{2}\tan\frac{\alpha}{2} = 1$. Multiplying both sides of this equality by $\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$ and then some simplifying give us the equality $\cos\frac{\alpha+\beta+\gamma}{2} = 0$. Thus, the condition $\alpha + \beta + \gamma = \pi$ is in this case equivalent to the condition $xy + yz + zx = 1$.

Bearing in mind that $\alpha + \beta + \gamma = \pi$ we express the left-hand side of (1) in another way, using trigonometric identities. We get

$\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 2\sin(\alpha+\beta)\cos(\alpha-\beta) + \sin 2\gamma = 2\sin\gamma\cos(\alpha-\beta) + 2\sin\gamma\cos\gamma = 2\sin\gamma(\cos(\alpha-\beta)+\cos\gamma) = 2\sin\gamma(\cos(\alpha-\beta)-\cos(\alpha+\beta)) = 2\sin\gamma(\cos\alpha\cos\beta+\sin\alpha\sin\beta- \cos\alpha\cos\beta + \sin\alpha\sin\beta) = 4\sin\alpha\sin\beta\sin\gamma = 32\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$.

Then we treat the right-hand side of (1) in a similar way and we get

$\sin\alpha + \sin\beta + \sin\gamma = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + \sin(\alpha+\beta) = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha+\beta}{2} = 2\sin\frac{\alpha+\beta}{2}\left(\cos\frac{\alpha-\beta}{2} + \cos\frac{\alpha+\beta}{2}\right) = 2\sin\frac{\alpha+\beta}{2}\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2} + \cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right) = 4\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha}{2}\cos\frac{\beta}{2} = 4\sin\left(\frac{\pi}{2} - \frac{\gamma}{2}\right)\cos\frac{\alpha}{2}\cos\frac{\beta}{2} = 4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$.

Hence, our inequality to be proven becomes

$32\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} \le 4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$, which, after dividing both sides of the inequality by $4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$ simplyfies to

(2) $\quad 8\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} \le 1.$

Now, since we have $0 < \alpha, \beta, \gamma < \pi$, we consider the function $f(x) = \sin x$. We note that $f''(x) = -\sin x < 0$ for $0 < x < \pi$. Then we first apply the AM-GM inequality to the left-hand side of (2) and after that Jensen's inequality (remembering that Jensen's inequality in this case is reversed). We find that

$8\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} \le 8\left(\frac{1}{3}\sin\frac{\alpha}{2} + \frac{1}{3}\sin\frac{\beta}{2} + \frac{1}{3}\sin\frac{\gamma}{2}\right)^3 \le 8\sin^3\left(\frac{1}{3}\cdot\frac{\alpha}{2} + \frac{1}{3}\cdot\frac{\beta}{2} + \frac{1}{3}\cdot\frac{\gamma}{2}\right) = 8\sin^3\left(\frac{\pi}{6}\right) = 1.$

PROBLEMS 2

41. Prove the inequality $\frac{a^n}{b+c} + \frac{b^n}{c+a} + \frac{c^n}{a+b} \ge \frac{a^{n-1}+b^{n-1}+c^{n-1}}{2}$.

42. Let $x_1, x_2, ..., x_n$ be positive real numbers. Prove that

$$x_1^{x_1}x_2^{x_2}...x_n^{x_n} \geq (x_1x_2...x_n)^{\frac{x_1+x_2+...+x_n}{n}}.$$

43. Prove that, for any real numbers x, y, $-\frac{1}{2} \leq \frac{(x+y)(1-xy)}{(1+x^2)(1+y^2)} \leq \frac{1}{2}$.

44. (IMO, 1995) Let a, b and c be positive real numbers such that $abc = 1$. Prove that $\frac{1}{a^3(b+c)} + \frac{1}{b^3(a+c)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}$.

45. (Moldavian MO, 1996) Let a, b and c be positive integers such that $a^2+b^2-ab=c^2$. Prove that $(a-c)(b-c) \leq 0$.

46. (Ireland, 2000) Let x and y be nonnegative real numbers such that $x+y=2$. Prove that $x^2y^2(x^2+y^2) \leq 2$.

47. (Thailand, 1991) Let a, b, c and d be positive real numbers such that $ab+bc+cd+da=1$. Prove that $\frac{a^3}{b+c+d} + \frac{b^3}{c+d+a} + \frac{c^3}{d+a+b} + \frac{d^3}{a+b+c} \geq \frac{1}{3}$.

48. (Greece, 1987) Let a, b and c be positive real numbers. Show that for every integer $n \geq 1$, $\frac{a^n}{b+c} + \frac{b^n}{c+a} + \frac{c^n}{a+b} \geq \left(\frac{2}{3}\right)^{n-2}\left(\frac{a+b+c}{2}\right)^{n-1}$ holds.

49. (Belarus, 1993) Let $x_1, x_2, ..., x_n$ $(n \geq 2)$ be nonnegative numbers such that $x_1+x_2+...+x_n = 1$. Prove that $\sum_{i=1}^{n} \sqrt{x_i(1-x_i)} \leq \sqrt{n-1}$.

50. (Hungary, 1985) Let a, b, c and d be positive real numbers. Prove that $\frac{a}{b+c} + \frac{b}{c+d} + \frac{c}{d+a} + \frac{d}{a+b} \geq 2$.

51. (Leningrad, 1981) Let a, b and c be real numbers such that $0 \leq a, b, c \leq 1$. Prove that $\sqrt{a(1-b)(1-c)} + \sqrt{b(1-a)(1-c)} + \sqrt{c(1-a)(1-b)} \leq 1 + \sqrt{abc}$.

52. (Romania, 2004) Find all positive real numbers a, b, c which satisfy the inequality
$4(ab+bc+ca)-1 \geq a^2+b^2+c^2 \geq 3(a^3+b^3+c^3)$.

53. (Romania, 2004) Let a, b and c be real numbers such that $a^2+b^2+c^2=3$. Prove that $|a|+|b|+|c|-abc \leq 4$.

54. (Estonia, 2004) Let a, b and c be positive real numbers such that $a^2+b^2+c^2=3$. Prove that $\frac{1}{1+2ab} + \frac{1}{1+2bc} + \frac{1}{1+2ca} \geq 1$.

55. (Austria, 2004) Let a, b, c and d be real numbers. Prove that
$a^6 + b^6 + c^6 + d^6 - 6abcd \geq -2$.

56. (Ireland, 2004) Let a and b be nonnegative real numbers. Prove that
$\sqrt{2}(\sqrt{a(a+b)^3} + b\sqrt{a^2+b^2}) \leq 3(a^2+b^2)$, with equality if and only if $a = b$.

57. (New Zealand, 2004) Let x_1, x_2, y_1, y_2 be positive real numbers. Prove that
$\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} \geq \frac{(x_1+x_2)^2}{y_1+y_2}$.

58. Let a, b and c be real numbers. Prove that
$\sqrt{a^2+(1-b)^2} + \sqrt{b^2+(1-c)^2} + \sqrt{c^2+(1-a)^2} \geq \frac{3\sqrt{2}}{2}$.

59. Let a, b and c be positive real numbers such that $abc = 1$. Prove that
$\frac{b+c}{\sqrt{a}} + \frac{c+a}{\sqrt{b}} + \frac{a+b}{\sqrt{c}} \geq \sqrt{a} + \sqrt{b} + \sqrt{c} + 3$.

60. Let a, b and c be positive real numbers. Prove that
$\frac{a}{(b+c)^2} + \frac{b}{(c+a)^2} + \frac{c}{(a+b)^2} \geq \frac{9}{4(a+b+c)}$.

61. (Juniors Balkan MO, 2002) Let a, b and c be positive real numbers such that $abc = 2$. Prove that $a^3 + b^3 + c^3 \geq a\sqrt{b+c} + b\sqrt{c+a} + c\sqrt{a+b}$.

62. Let a, b and c be positive real numbers such that $abc \leq 1$. Prove that
$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq a + b + c$.

63. (Juniors Balkan MO, 2002) Let a, b and c be positive real numbers. Prove that $\frac{a^3}{b^2} + \frac{b^3}{c^2} + \frac{c^3}{a^2} \geq \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}$.

64. Let x, y and z be positive real numbers such that $x^2 + y^2 + z^2 + 2xyz = 1$. Prove that $x + y + z \leq \frac{3}{2}$.

65. (Juniors Balkan MO, 2003) Let x, y and z be real numbers such that
$x, y, z > -1$. Prove that $\frac{1+x^2}{1+y+z^2} + \frac{1+y^2}{1+z+x^2} + \frac{1+z^2}{1+x+y^2} \geq 2$.

66. Let a, b and c be positive real numbers such that $a + b + c = 1$. Prove that
$\frac{a^2+b}{b+c} + \frac{b^2+c}{c+a} + \frac{c^2+a}{a+b} \geq 2$.

67. (IMO, 1987) Let x, y and z be real numbers such that $x^2 + y^2 + z^2 = 2$. Prove that $x + y + z \leq xyz + 2$.

68. Let x, y, z and α be positive real numbers such that $xyz = 1$ and $\alpha \geq 1$.
Prove that $\dfrac{x^\alpha}{y+z} + \dfrac{y^\alpha}{z+x} + \dfrac{z^\alpha}{x+y} \geq \dfrac{3}{2}$.

69. Let a, b, c and d be positive real numbers. Prove that
$(a+b)^3(b+c)^3(c+d)^3(d+a)^3 \geq 16a^2b^2c^2d^2(a+b+c+d)^4$.

70. Let a, b and c be positive real numbers such that $a + b + c = 1$. Prove that
$(a^2+b^2)(b^2+c^2)(c^2+a^2) \geq 8(a^2b^2 + b^2c^2 + c^2a^2)^2$.

ONE MORE USEFUL INEQUALITY

The AM-GM inequality can easily be generalized to matrices with nonnegative elements.

Let $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$ be a matrix with nonnegative elements. Let G_1, G_2, ..., G_m denote the geometric means of the rows of A and let A_1, A_2, ..., A_n be the arithmetic means of the columns of A. Suppose that all the numbers A_1, A_2, ..., A_n are positive.

Consider one of the rows of A, let's say the first row. After dividing a_{1k} by A_k $(k = 1, 2, ..., n)$, we get a list of n numbers: $\dfrac{a_{11}}{A_1}, \dfrac{a_{12}}{A_2}, ..., \dfrac{a_{1n}}{A_n}$.

Applying now the AM-GM inequality to those numbers yields

$$\frac{1}{n}\sum_{k=1}^{n}\frac{a_{1k}}{A_k} \geq \sqrt[n]{\frac{a_{11}a_{12}...a_{1n}}{A_1A_2...A_n}} = \frac{G_1}{G(A_1, A_2, ..., A_n)} \ ,$$

where $G(A_1, A_2, ..., A_n)$ denotes the geometric mean of A_1, A_2, ..., A_n.

After repeating this procedure for all rows of A, we may add all m inequalities. This will give

$$\frac{1}{n}\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{a_{jk}}{A_k} \geq \sum_{j=1}^{m}\frac{G_j}{G(A_1, A_2, ..., A_n)} \ .$$

After rearranging the terms on the left-hand side we will get

$$\frac{1}{n}\sum_{k=1}^{n}\frac{\sum_{j=1}^{m} a_{jk}}{A_k} \geq \frac{\sum_{j=1}^{m} G_j}{G(A_1, A_2, ..., A_n)} \ ,$$

which is equivalent to $\frac{1}{n}\sum_{k=1}^{n} m \geq \frac{\sum_{j=1}^{m} G_j}{G(A_1, A_2, ..., A_n)}$, i.e. $1 \geq \frac{\frac{1}{m}\sum_{j=1}^{m} G_j}{G(A_1, A_2, ..., A_n)}$. If we let $A(G_1, G_2, ..., G_m)$ be the arithmetic mean of G_1, G_2, ..., G_m, then the obtained inequality can be written as $G(A_1, A_2, ..., A_n) \geq A(G_1, G_2, ..., G_m)$. It should be obvious that this inequality is valid even if we allow some of the A_k to be 0. Hence we have proved the following, powerful generalization of the AM-GM inequality:

Theorem 4. If A is a matrix with m rows, n columns and nonnegative elements, let G_1, G_2, ..., G_m denote the geometric means of the rows of A and let A_1, A_2, ..., A_n be the arithmetic means of the columns of A. Moreover, let $G(A_1, A_2, ..., A_n)$ denote the geometric mean of A_1, A_2, ..., A_n and let $A(G_1, G_2, ..., G_m)$ be the arithmetic mean of G_1, G_2, ..., G_m. Then

$$G(A_1, A_2, ..., A_n) \geq A(G_1, G_2, ..., G_m) .$$

□

We give now some applications of this theorem.

1). Applying the theorem to the matrix $A = \begin{pmatrix} a_1^2 & b_1^2 \\ a_2^2 & b_2^2 \\ ... & ... \\ a_m^2 & b_m^2 \end{pmatrix}$, and then using the triangle inequality $\sum_{k=1}^{m} |x_k| \geq |\sum_{k=1}^{m} x_k|$ yields $\sqrt{\Big(\sum_{k=1}^{m} a_k^2\Big)\Big(\sum_{k=1}^{m} b_k^2\Big)} \geq \sum_{k=1}^{m} |a_k b_k| \geq \Big|\sum_{k=1}^{m} a_k b_k\Big|$. Squaring both sides gives the Cauchy-Schwarz inequalitiy.

2). Let $c_1, c_2, ..., c_m$ be m positive rational numbers with the sum 1. Let M be the smallest common denominator of c_1, c_2, ..., c_m and put $c_k = \frac{d_k}{M}$, for $k = 1, 2, ..., m$. Then, of course, $\sum_{k=1}^{m} d_k = M$.

Suppose we have m sequences of positive real numbers: $\big(a_{11}, a_{12}, ..., a_{1n}\big)$, $\big(a_{21}, a_{22}, ..., a_{2n}\big)$, ..., $\big(a_{m1}, a_{m2}, ..., a_{mn}\big)$. Consider the matrix A with n rows and $d_1 + d_2 + ... + d_m$ columns, where each of the first d_1 columns equals the sequence $\big(a_{11}, a_{12}, ..., a_{1n}\big)$, each of the next d_2 columns equals the sequence $\big(a_{21}, a_{22}, ..., a_{2n}\big)$, and so on.

Applying now the theorem 4 to the matrix A we get the (generalized) Hölder's inequality:

$$\Big(\sum_{k=1}^{n} a_{1k}\Big)^{c_1}\Big(\sum_{k=1}^{n} a_{2k}\Big)^{c_2}...\Big(\sum_{k=1}^{n} a_{mk}\Big)^{c_m} \geq \sum_{k=1}^{n} a_{1k}^{c_1} a_{2k}^{c_2} ... a_{mk}^{c_m}.$$

3). Suppose now that n, k and m are positive integers, $k \leq m$. Let a_1, a_n,..., a_n be nonnegative real numbers. Consider the matrix A with n rows and m columns:

$$A = \begin{pmatrix} a_1^m & a_1^m & ... & a_1^m & 1 & 1 & ... & 1 \\ a_2^m & a_2^m & ... & a_2^m & 1 & 1 & ... & 1 \\ ... & ... & ... & ... & ... & ... & ... & ... \\ a_n^m & a_n^m & ... & a_n^m & 1 & 1 & ... & 1 \end{pmatrix},$$

where the first k columns are identical and the elements in the remaining $m-k$ columns are only 1's.

The theorem 4 applied to this matrix yields the Power Mean inequality

$$\left(\frac{1}{n}(a_1^m + a_2^m + \cdots a_n^m)\right)^k \geq \left(\frac{1}{n}(a_1^k + a_2^k + \cdots a_n^k)\right)^m$$

4). Several exercises from the list of problems can be directly solved by the use of the theorem 4, for a suitable choice of the matrix A. For example, taking $A = \begin{pmatrix} 1 & 1 & ... & 1 \\ x_1 & x_2 & ... & x_n \end{pmatrix}$ and applying the theorem gives an immediate solution to problems 20 and 36.

RECENT PROBLEMS

Problem 71. (IMO 2005) Let x, y, z be positive real numbers such that $xyz \geq 1$. Prove that $\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0.$

Solution. This problem was considered by the jury of the IMO 2005 as a hard and was the third problem on the exam. The standard solution is rather long. One may reduce the problem to the case where $xyz = 1$ only, then get rid of the denominators, homogenize and finally make use of the Muirhead's inequality. The procedure is rather standard although the calculations are quite tedious. However, during the competition one of the students came up with the following, very short and extremely elegant solution.

The annoying part of the left hand side is that the denominators have terms with different powers. In order to get rid of this obstacle one may consider the following inequality:

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} \geq \frac{x^2(x^3 - 1)}{x^5 + (y^2 + z^2)x^3}.$$

This inequality is obviously true if $x \geq 1$. It remains true even for $0 < x < 1$, since in this case the nominators are negative. Hence

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} \geq \frac{x^2 - \frac{1}{x}}{x^2 + y^2 + z^2}.$$

Applying the same argument to the other two terms of the left-hand side of the given inequality, we find out that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq \frac{1}{x^2 + y^2 + z^2}\Big(x^2 - \frac{1}{x} + y^2 - \frac{1}{y} + z^2 - \frac{1}{z}\Big).$$

What now remains to show is that $x^2 - \frac{1}{x} + y^2 - \frac{1}{y} + z^2 - \frac{1}{z} \geq 0$, i.e. that $x^2 + y^2 + z^2 \geq \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$, for positive reals x, y, z with $xyz \geq 1$. This however is a very simple task. $\square$