COMBINATORIAL METHODS IN EDUCATION AND TRAINING

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The history of combinatorial problems in competitions goes back to the second haungarian Eotvos/Kurschak contest, 1895. Combinatorial problems were also always present from the very start of the Moscow Mathematical Olympiads, 1935. In the IMO, the first such problem occurred in the fifth competition, 1963. Nevertheless it was only during the last few years that combinatorial problems established themselves as an integral part of the IMO.

I see combinatorics as this branch of competitional mathematics where one has least use of knowing a large number of theorems and theories. Almost each problem leads to some new and unknown situations and demands fantasy and creativity. The only way to train a student in problems in combinatorics and to get him/her used to the very special approaches which such problems demands is to let the student solve on his own as many problems as possible. Nevertheless, there are some standard methods of attacking combinatorial questions and those methods should be a ”must” in training future competitors.

Combinatorial problems may roughly be divided in five categories:

1. Find whether there exist at least one object of the given kind.
2. Indicate at least one such object.
3. Find the number of such objects.
4. Give an algorithm for constructing all such objects.
5. Find among all objects an object which is optimal in some way.

There are several methods for solving problems from those five groups, both constructive and non-constrctive. For example, problems from the second category demands constructive argument while problems from the fifth category demands a constructive methods for finding an object and then, most often, non-constructive argument for proving optimality.

The solving methods themselves may be divided in two main categories which may roughly be described as:

1. Find the properties of the whole sets given the properties of its elements (here we find for example the method of mathematical induction).
2. From the properties of a set find some information about the individual elements (the typical method here is the Box Principle).

In the sequel I will list some most elementary ways of approaching combinatorial problems and illustrate those methods with several examples from the previous competitions.

1. Looking at some special cases; getting partial results

When one approaches a standard mathematical problem during a competition the usual way is to look through all the repertoire of methods of solutions, tricks and theorems one can remember from the previous training. But often, especially often when it comes to combinatorics, this method doesn't lead anyway: problem in question seems to be totally different from all the problems one have solved before and this is, in my opinion, quite a nice and stimulating situation. It means the problem is rather interesting and one may learn something new by solving it.

The good way to start with is to do some experiments hoping to discover some patterns or strategies. One can try to solve the problem for some small values of *n* or one can look at the smaller parts of the given problem constellation in order to get at least some partial results. It is though important to keep a track of all those partial results for very often the total solution is just a right combination of those.

As an example of such strategy consider a problem from the IMO 1970.

100 points are given in the plain such that no three of them are collinear. Show that at most 70% of triangles having their vertices in those points are acute triangles.

Solution: The best approach towards the solution is to look first at the small cases with 4 and 5 points. The convex hull of 4 points may be a quadrilateral (with at least one non-acute angle) giving at most 3 acute triangles (75%). If the convex hull is a triangle at least one of the triangles having the interior point as a vertex is obtuse. This gives again at most 3 acute triangles.
For 5 points can we choose four points in 5 ways, each four points giving at most 3 acute triangles. Every one of those triangles is counted twice giving in total at most [5·3/2]=7 acute triangles (70%).
By now we should have an idea how to estimate the total number of acute triangles, given 100 points. There are $\binom{100}{5}$ groups of 5 points, each having at most 7 acute triangles, and each triangle being counted $\binom{97}{2}$ times. This gives at most $7\cdot\binom{100}{5}/\binom{97}{2}$ acute triangles in total. This number, divided by the total number of triangles, $\binom{100}{3}$, is $\frac{7}{10}$.

Similar problem of the same kind comes from the IMO 1969.

There are n points in the plane, n>4, such that no three lie on the same line. Prove that one can find at least $\binom{n-3}{2}$ *convex quadrilaterals having their vertices in given points.*

Another easy strategy in solving combinatorial problems involving proving some property is just to start a construction of an object without such property and see "how far" one can get and what causes the eventual stop. As an example of such "naive" approach consider a problem from the IMO 1979.

We are given a prism with pentagons $A_1A_2A_3A_4A_5$ and $B_1B_2B_3B_4B_5$ as top and bottom faces. Each side of the two pentagons and each of the line segments A_iB_j for all i,j = 1,2,3,4,5 is coloured either green or yellow. Every triangle whose vertices are vertices of the prism and whose sides has all been coloured has two sides of different colour. Show that all ten sides of the top and the bottom faces of the prism are of the same colour.

Solution: Suppose the contrary. Then there are two cases to consider: (1) there are two different colours adjacent to a vertex of one of the pentagons, or (2) the sides of one pentagon are all green while the sides of the other are all yellow.

In case (1) we may assume A_1A_2 is green while A_2A_3 is yellow. The natural way now is to experiment with the colours of A_2B_j (j=1,2,3,4,5), and maybe the edges A_1B_j, A_3B_j, trying to avoid monochromatic triangles. One easily find the key observation that two consecutive edges of A_2B_j must be of different colours and then the sides of the B-pentagon must be monochromatic. Then we get stuck when trying to colour edges from B-pentagon to A_1 and A_3. This is basically the solution in this case and all what remains is to nicely write down the argument.

The case (2) is treated similarly.

Another example, which is easiest solved by the same kind of experimental approach is a problem from the Nordic Mathematical Competition 1987.

Nine journalists meet at the conference. Each one speaks at most three languages and each pair of journalists speaks a common language. Show that at least five of those nine journalists speak a common language.

2. Counting principles

Many combinatorial problems may be easily solved by counting and in a smart way comparing results of counting. It is though important to find out what is needed to be counted and how the counting should be done.

The Dirichlets Box Principle (Pigeonhole Principle) is a very useful tool in various situation. Since we often encounter situations when there are several possible ways of applying the Principle, we may need several different attempts before we find the right way to solve the problem. What is needed to capture in counting is enough of the specific problem conditions and demands, without, of course, getting into some impossible calculations. Consider the following problem:

A baker has an access to at least 3 spices. For baking 10 different loaves of bread he uses more than half of his spices for each loaf. Show that there are three spices such that each loaf contains at least one of them.

Solution: One natural thing to count would be the number a_k of loaves containing the specific spice k (k=1,2,…,n, where n is the number of spices). Since there is not much one can say about each and one of the numbers a_k (it may be all between 0 and 10) one may instead try to count their sum.

Now, what do we get the when adding all those numbers? The sum seems to be the same as the number of spices used in each loaf added together. If we then by c_b denote the number of spices in the loaf b then we can write it down as $a_1+a_2+\ldots+a_n = c_1+c_2+\ldots+c_{10}$. In this step we may have use of the condition of number of spices per loaf, given in the problem, which is $c_b > n/2$ for all b. This implies that the sum $a_1+a_2+\ldots+a_n$ is greater than $10 \cdot n/2 = 5n$.

(The argument above uses yet another, important combinatorial counting method: the double-counting where the same sum counts in two different ways. It's like counting the sum of elements of a matrix by adding row-wise or column-wise.)

The Box Principle implies now that at least one of the numbers a_k is greater than 5. In other words, at least one of the spices is contained in at least 6 loaves! This however doesn't solve the problem yet.

Another consequence of the inequality $c_b > n/2$ is that for each pair of loaves there is at least on spice included in both of them (intuitively easy but needs of course a formal proof).

The two facts already proven lead now straight to the complete solution: Take one spice, k_1, which is included in 6 loaves. Then we have 4 loaves left, let's say b_1, b_2, b_3 and b_4. Then wa have a spice k_2 which is in b_1 and b_2 and yet another, k_3, common for b_3 and b_4. The spices k_1, k_2, and k_3 fulfill the requirements of the problem.

Two example of problems with similar solving strategies (double-counting and the Box Principle) are:

Each of 144 points (x,y) with integer coordinates, 1 ≤ x, y ≤ 12, is painted red, white or blue. Show that there exists a rectangle with sides parallel with the axis and with all vertices of the same colour. (Sweden, 1982).

Let M be a set of 10 distinct positive integers, all of them being two-digits numbers. Show that it is possible to find in M two disjoint subsets such that the sum of elements in each subset are equal. (IMO, 1972).

Now I'd like to turn the attention to one more very useful tool for solving combinatorial problems, the Inclusion-Exclusion Principle.

Suppose we have a set of N objects such that each one of them may have some of the properties $E_1, E_2, ..., E_n$. Suppose N_i is the number of objects with the property E_i, $N_{i_1 i_2}$ is the number of objects with the properties E_{i_1} and E_{i_2}, and so on. Then, the number A of objects having none of the properties $E_1, E_2, ..., E_n$ is

$$A = N - \sum_i N_i + \sum_{i_1, i_2} N_{i_1 i_2} - \sum_{i_1, i_2, i_3} N_{i_1 i_2 i_3} + ... + (-1)^n \sum_{i_1, ..., i_n} N_{i_1 i_2 ... i_n} .$$

Moreover, following inequalities are valid:

$$A \geq N - \sum_i N_i,$$

$$A \leq N - \sum_i N_i + \sum_{i_1, i_2} N_{i_1 i_2},$$

$$A \leq N - \sum_i N_i + \sum_{i_1, i_2} N_{i_1 i_2} - \sum_{i_1, i_2, i_3} N_{i_1 i_2 i_3},$$

and so on.

A rather easy problem, where one can use the simplest form of the inclusion-exclusion principle, comes from the Tournament of Towns, 1994.

There are 20 children in a village, each two of them having a common grandfather. Show that at least 14 of those children have a common grandfather.

Solution: Let a be one of the children and let A and B be his/hers two grandfathers. Let X be the set of children having both A and B as grandfathers, Y – children sharing with a only A as a grandfather, and Z - children sharing with a only B as a grandfather. A child from Y have only one more grandfather and sharing a grandfather with every child from Z implies that there is only one more grandfather, C, the one being a common grandfather for all children from Y and Z.
Let now A', B' and C' denote the sets of grandchildren of A, B and C respectively. Then we have $20 = |A'+B'+C'| = |A'|+|B'|+|C'| - (|A'\cap B'|+|A'\cap C'|+|B'\cap C'|) = |A'|+|B'|+|C'| - (|X|+|Y|+|Z|) = |A'|+|B'|+|C'| - 20$. This means that $|A'|+|B'|+|C'| = 40$, which implies that at least one of the sets A', B' and C' has at least 14 elements.

Another example of a problem, which is easily solved by this method, is a question from IMO 1989:

Let n be a positive integer. A permutation $(x_1, x_2, \ldots, x_{2n})$ of the set $\{1, 2, \ldots, 2n\}$ has the property P if $|x_i - x_{i+1}|=n$ for at least one i. Show that for every n, $n\geq 1$, the number of permutations with the property P is greater than the number of permutation which don't have it.

A solution, based on the inclusion-exclusion principle, may use the inequality $A \leq N - \sum_{i=1}^{2n-1} N_i + \sum_{1\leq i<j\leq 2n-1} N_{ij}$, where A is the number of permutations which do not have the property P, N_i is the number of permutations for which $|x_i - x_{i+1}|=n$, and N_{ij} is the number of permutations for which $|x_i - x_{i+1}|=n$ and $|x_j - x_{j+1}|=n$.
It is obvious that $|N| = (2n)!$, but little less immediate that $|N_i| = 2n(2n-2)!$ (the numbers x_i and x_{i+1} can be chosen in $2n$ ways while all the other numbers can be permuted in $(2n-2)!$ ways), and $|N_{ij}| = 8\cdot\binom{n}{2}\cdot(2n-4)!$ (numbers k and m, both $\leq n$, together with $k+n$ and $m+n$ are to be put in places i, $i+1$, j, and $j+1$. Those numbers can be chosen in $\binom{n}{2}$ ways, then distributed in 8 ways, while all the other numbers may be permuted in $(2n-4)!$ ways). Moreover there are $2n-1$

permutations N_i and $\binom{2n-1}{2}-(2n-2)$ permutations N_{ij} (for all values i, j). Adding together the right-hand side we get $(2n)!\frac{n-1}{2n-1}$, which is clearly < (2n)!/2 (for n at least 2. For n=1 the conclusion is obvious).

Consequently, the number of permutations with the property P is greater than the half of the number of all permutations and the solution is complete.

Still another example of a problem where one should use the same principle is the following:

A "word" of length n means here a sequence of n, n≥1, symbols from the "alphabet" {A, B, C}. Show that the number of words that nowhere include a sub-word (consecutive symbols) ABC is

$$3^n - \binom{n-2}{1}\cdot 3^{n-3} + \binom{n-4}{2}\cdot 3^{n-6} - \binom{n-3}{3}\cdot 3^{n-9} + \ldots$$

3. Method of extremal elements

In the elementary mathematics the method is used in the following way: If we have to study some set A, we focus on the element b of A which is extremal in some sense (this condition is often a definition of b). After that, analyzing the connection between b and other elements of A, we obtain some useful information about b or about the whole set A.

The method is widely used in different branches of mathematics. The combinatorial use may be exemplified by the following (rather easy) problem:

In the country of Oz there is a well developed but lousy organized network of flight connections: Between each pair of cities there is a flight connection although only in one direction. Show that there is a city in Oz from which one can reach any other city with at most one change of planes.

Solution: The natural thing in such situations is to look at the city with the maximal number of outgoing connections. Let's call this city (or, one of those cities) for A and let B_1, B_2, …, B_k be all those cities which can be reached directly from A. Denote by C_1, C_2, …, C_m all the remaining cities of Oz.

We try to show that the city A solves our problem. For suppose, in the contrary, that one of the C-cities, let's assume C_1, cannot be reached directly from any one of the B-cities. Then there is a flight connection from C_1 to each one of the B-cities (we know that there is an one-direction connection between each pair of cities of Oz). Moreover, since there is a direct flight from C_1 to A, then the number of outgoing flights from C_1 is greater that that the number of outgoing flights from A. This contradicts the assumed maximality.

Another example comes from the Hungarian Mathematical Olympiad 1973.

Suppose s_n is the number of space parts into which n planes, $n \geq 5$, in general position divide R^3. Show that at least $(2n-3)/4$ of those parts are tetrahedra.

Solution: Let Π be one of these planes. It divides the space in two parts, S_1 and S_2. The vertices (a vertex is a point of intersection of three planes) not lying on Π are either on on both sides of Π, or only on one side, let's say in S_1. In the first case we use the exteremal principle and choose a vertex A having the minimum distance to the plane Π. Since A is the intersection of planes Π_1, Π_2 and Π_3, so, together with the plane Π, we have a tetrahedron $ABCD$ (if any of the remaining n-4 planes cuts this tetrahedron then it must cut at least one of the edges AB, AC, or AD giving a vertex closer to Π than A, hence a contradiction). A similar argument gives us another tetrahedron on the other side of Π.

Since the argument is valid for every plane we may have found the reason for the term $2n$ in the formula. The number 4 in the denominator is expected, since each tetrahedron is counted 4 times, once for each face.

Now we have to find the reason for –3 in the numerator. There is still one case we didn't consider: the case when all the remaining vertices lie on the side S_1.

It's not difficult to show that there may be at most three such planes. For suppose there are four such planes, Π_1, Π_2, Π_3 and Π_4. They define a tetrahedron $ABCD$. Since there exists at least one more plane Π, it must cut at least on of the edges of the tetrahedron on its extension outside

ABCD. This gives a vertex being on the ”wrong” side of one of the planes Π_1, Π_2, Π_3 and Π_4. This contradiction means that there are at most 3 planes which generate only one tetrahedron each. This completes the proof of the desired formula.

One more, very easy, question, where looking at the extremal element quickly leads to the solution. It is a problem from IMO 1968.

Show that in any tetrahedron there is a vertex such that the edges incident with this vertex may form the sides of a triangle.

4. Method of invariants

In a few words, the Method of Invariants can be described in the following way: If there is a repeated process, try to find out what does not change.

An easy example from the Swedish Mathematical Competition 1970 is

The numbers from 1 to 500 are written on a piece of paper. I a single move you chose 2, 3, 4 or 5 numbers from the list, erase them and adjoin to the list the remainder of the sum of the chosen numbers when it is divided by 13. After a number of moves there are only two numbers left on the page. One of them is 102. What is the other?

Solution: It is easy to find a simple invariant. The sum of all numbers on the list modulo 13 doesn’t change after each move. Moreover, each new number is less than 13. The number x we are looking for satisfies then $x+102 \equiv 1+2+\ldots+500 \pmod{13}$, which gives $x=10$.

Slightly more difficult is a problem from the Leningrad Mathematical Olympiad, 1988.

Consider any binary word $W = a_1a_2...a_n$. One can transform it by inserting or deleting any word of type BBB, where B is a binary word. Is there a sequence of transformations, which will produce the word 10 when starting with the word 01?

Solution: Three repetitions of the word B should suggest looking at the sum of some linear combination of the "letters", counted modulo 3. One possible such sum is $f(W) = a_1 + 2a_2 + \ldots + na_n$. One can show that the transformation doesn't change the value of the function f and, since $f(01)=2$ and $f(10)=1$, the answer is: no.

Generally, when applying the Method of Invariance we are looking for a function f(object) such that f doesn't change its value after performing one transformation. The invariant itself may be numerical but even "qualitative", like parity, orientation or congruence.

Instead of an invariant can we sometimes find a half-invariant function for a given problem. It is a function f which is monotone, i.e. f increases or decreases after performing a transformation. If, moreover, this function can only take finitely many values then we may have found a great tool for proving that a specific process will eventually come to a stop. Consider the following example:

The vertices of an n-gon are labeled by real numbers $a_1, a_2, \ldots, a_n$. Let a, b, c and d be four successive labels. If $(a-d)(b-c)<0$, then we may switch b with c. Can the labels be arranged in such a way that this process will never end?

Solution: Suppose T_1 is a distribution of labels in order $a_1, a_2, \ldots, a_n$. The inequality $(a-d)(b-c)<0$ can be rewritten as a$b + cd < ac + bd$. Let's introduce the function $f(T_1) = a_1 \cdot a_2 + \ldots + a_{n-1} \cdot a_n + a_n \cdot a_1$. Since the switching of labels replaces $ab+bc+cd$ with $ac+cb+bd$, it's easy to see that f is an increasing function. At the same time f may only have a finitely many values. Therefore, the process cannot be repeated infinitely many times.

One more example comes from the Swedish Mathematical Competition 1989:

4n points ($n \geq 1$) are chosen on a circumference of a circle. The points are painted alternatively yellow and blue. One partitions the set of the yellow points in pairs and between points of each pair one draws a yellow line. Similarly, one partitions the set of the blue points in pairs and between points of each pair one draws a blue line. Suppose that no three lines pass through the same point inside the circle. Show that there are at least n points of intersection between blue and yellow lines.

Solution: Let T_1 be the given configuration and let $f(T_1)$ be the number of points of intersection between blue and yellow lines. Suppose that two lines of the same colour, let's assume blue lines, AB and CD meet each other. Then we can replace them with two other non-crossing blue lines AC and BD. It's not difficult to show that for this new configuration, T_2, the inequality $f(T_2) \leq f(T_1)$ holds. Moreover, the number of points within the circle, where two lines of the same colour meet decreases by at least 1. By repeating the replacement procedure we will end with the configuration T_k where there is no crossing points between two lines of the same colour.
The final, easy step of the solution is to prove that $f(T_k) \geq n$.

Finally, the last example:

The following transformations are permitted with the quadratic polynomial Ax^2+Bx+C: (1) the coefficients A and C may be switched, and (2) x may be substituted by $x+D$ for any choice of a real constant D. By repeating these operations, can the polynomial x^2-x-1 be transformed to x^2-x-2?

(An idea: try the discriminant!)

5. Method of colouring

Several combinatorial problems may be solved very elegant by partitioning the underlying set into a finite number of subsets. One often speaks about colouring the elements in a number of colours. The colouring arguments are sometimes strikingly pretty and are especially useful when dealing with different types of combinatorial covering problems. A typical example may be the following question:

We are given 21 I-trominoes (3 unit squares in a row) and one unit square. With those 22 tiles it is possible to construct an 8x8 square. Find all the positions where the unit square may be placed.

Solution: Consider the 8x8 square and colour the unit squares in each row from the left to the right in three colours, 1, 2 and 3. Colour the first row as 12312312, the second as 31231231, the third as 23123123, the next row as the first one, and so on. Now we need to make two crucial observations: (1) we have painted 22 unit squares in colour 1, 21 in colour 2 and 21 in colour 3. Moreover, (2) every 1x3 tromino will have its squares painted in three different colours. The conclusion we may draw is that the unit square must occupy a place painted with colour 1.
If we now repaint the squares again with the same three colours but doing it from the right to the left, then we may of course make the very same two observations.
Then we only notice that there are exactly four squares, which were painted by colour 1 in both colourings, squares where the third and the fifth rows meet the third and fifth column.
Because of symmetry it only remains to construct the 8x8 square using our 22 tiles and placing the unit square in one of those four indicated squares. This can be easily done.

Another example comes from the Austrian Mathematical Olympiad, 1989.

A 23x23 square is completely tiled by 1x1, 2x2 and 3x3 tiles. What minimum number of 1x1 tiles is needed?

Solution: Suppose no 1x1 tiles are needed. Colour the *rows* of the 23x23 square alternatively in black and white, starting with a black row. Then the number of black unit squares will be 23 more than the number of white unit squares. One 2x2 tile covers always 2 black and 2 white squares, while the difference between the numbers of white and black squares covered by one 3x3 tile is always 3. In total, the difference between the numbers of black and white unit squares

covered by 2x2 and 3x3 tiles is divisible by 3. Since 23 is not divisible by 3, the tiling is not possible.
However the tiling is possible when using only one 1x1 tile. One may place it in the middle of the 23x23 square. The remaining part can be split in four 11x12 parts and it is easy now to cover each part with six 2x2 and twelve 3x3 tiles.

For which n≥2 there exists a closed knight's tour on a 4xn board?

(An idea: Colour the unit squares in four colours, using all colours in each column and having the order of colours in odd columns the reverse of the order of colours in even columns.)

6. Greedy algorithm

Some problems concerning existence of a specific object can be solved by a kind of a "naive", straightforward construction. One can just try to do "the best possible at the moment" and one only has to remember to keep the most basic precautions. The greedy algorithm is one of the best strategies in this respect; it leads surprisingly often to the desired results. Consider one problem from the IMO 1983.

Is it possible to choose 1983 distinct positive integers, all less than or equal to 10^5, no three of which are consecutive terms of an arithmetic progression?

Solution: Let's try with a sequence beginning with 1 and 2. Next number cannot be 3 but 4 and 5 may be added to the list. 6, 7, 8 and 9 are of course eliminated but 10 and 11, and then 14 and 15 will do fine.
By now we should be able to discover that having already constructed a sequence of 2^k numbers: 1, 2, 4, 5, …, a_k, we may easily add another set of 2^k numbers starting with $2a_k$ and repeating the previous pattern: $2a_k$, $2a_k+1$, $2a_k+3$, $2a_k+4$, …. We will then end up with 2^{k+1} numbers between 1 and $a_{k+1} = 3a_k – 1$.

One should of course write down a formal proof for the validity of this construction: Three numbers, $a<b<c$ from already constructed sequence cannot form an arithmetic progression. For suppose a is in the first part ($\leq a_k$) while b and c are in the second part. Then $2b \geq 4a_k > 3a_k - 1 + a_k \geq c + a$. If a and b are in the first part while c is in the second then $2b \leq 2a_k < 1 + 2a_k \leq a + c$. Since we have $a_1=2$ and $a_2=5$, we may continue until $a_{11}=88574$. This means that between 1 and 88574 we can have a sequence of $2^{11}=2048$ numbers, free from arithmetic progressions.

Another problem, inviting to a similar solution-strategy, comes from IMO 1991.

Suppose G is a connected graph with k edges. Prove that it is possible to label the edges 1, 2, ... , k in such a way that at each vertex, which belongs to two or more edges, the greatest common divisor of the integers labeling those edges is 1.

Solution: Start labeling in any vertex A. Take a path from A along unlabeled edges and label the edges consecutively 1, 2, 3, ... as the path is constructed. The procedure stops when reaching a vertex B without unlabeled edges. It's easy to see that all the vertices which we passed (including B) satisfy the condition of the problem (gcd at A is 1; gcd at all vertices between A and B is 1 as well because at each verex there are two edges labeled with consecutive numbers; finally, if B coincides with any of the previous vertices then gcd at B is 1, otherwise there is only one edge at B and there is no gcd to consider).
Now we can take any vertex C with an unlabeled edge and repeat the process. The same argument shows that all the new vertices on the new path have gcd 1. This labeling procedure may be repeated until all the edges are labeled.

A final, more difficult example comes from IMO 1986.

Given a finite set of points in the plane, each with integer coordinates, is it always possible to colour the points red or white so that for any straight line L parallel to one of the coordinate axes the difference (in absolute value) between the numbers of white and red points on L is not greater than 1?

7. Last topics

In the end I just mention another two very important combinatorial topics needed careful treatment and training: Graph Theory and Combinatorial Geometry.

Graph theory has been a source for many combinatorial problems in competitions; many others can be easily reformulated in graph theoretic terms. Therefore at least basic knowledge of graphs is another "must" in training future competitors. There is a very rich variety of this kind of problems in the literature from different competitions.

Combinatorial geometry, mainly configuration of points and other figures in the plane and in the space, makes also a frequent appearance in mathematical competitions. Several problems of this kind demand yet another approach: the notion of the convex hull. This very fruitful notion deserves, in my opinion, as much attention in the preparation of students as any of the other techniques discussed above.

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